2. Representation of Signals

Signals may be classified as predictable or as unpredictable, as analog or discrete and of finite or infinite duration. **Deterministic signals** are defined exactly as a function of time. They can be periodic (such as a sinewave or squarewave) or they can be an aperiodic “one shot” signal. Deterministic signals contain no information because their future is completely predictable by the receiver. They are easy to model and are useful since they can provide a reasonably accurate evaluation of communication system performance. **Stochastic signals**, on the other hand, are unpredictable and thus can communicate information. Although the time waveform of a stochastic signal is random, the signal power may be predictable. Examples of stochastic signals are thermal noise in electronic circuits, (i.e. “background hiss”) and information signals such as voice or music.

An **analog signal**, denoted \( x(t) \), is a *continuous function of time* and is uniquely determined for all \( t \). When a physical signal such as speech is converted to an electrical signal by a microphone, we have an electrical analog of the physical waveform. An equivalent **discrete-time signal**, denoted as \( x(kT) \), exists only at discrete instants. It is characterized by a sequence of values that exist at specific times, \( kT \), where \( k \) is an integer and \( T \) is normally a fixed time interval. On the other hand, a continuous time signal may be restricted to a set of **discrete amplitudes**. A signal that is discrete in both time and amplitude is referred to as a **digital signal**. Furthermore, these discrete digital signal amplitudes can be represented by a set of numbers (codes) and, as such, can be stored in a computer memory. Pulse code modulation (PCM) is an example of a digital signal. These categorizations are illustrated in Figure 2-1.

![Figure 2-1 Concept of Analog and Digital Signals](image-url)
An energy waveform has finite amplitude and it either exists for a finite duration or it decays to zero over time. An energy signal has finite energy but zero average power. A power waveform has finite amplitude and semi-infinite duration thus it has finite average power but semi-infinite energy. The signal is considered to be continuous over the observation period. Energy and power classifications are mutually exclusive; a signal must be one or the other. Periodic signals are classified as power signals.

2.1 SIGNAL POWER

**Instantaneous power** is the voltage-current product at a specific time, \( p(t) = v(t) i(t) \), while **average power** is \( P = \langle p(t) \rangle = \langle v(t) i(t) \rangle \) where \( \langle \cdot \rangle \) indicates time average. With **normalized power**, the load resistance is assumed to be 1 Ω, current is numerically equal to voltage and average normalized power is simply the time average of voltage squared, \( P_N = \langle v(t)^2 \rangle \). Root-mean-square (rms) voltage is the square root of normalized power, \( V_{rms} = \sqrt{P_N} \). For example, the 12 volt supply in an automobile has normalized power of 144 ‘watts’.

**Example 2.1** – Integrate to find the time average normalized power of \( v(t) = A \cos \omega_0 t \) (period \( T_0 = \frac{2\pi}{\omega_0} \)).

**Solution:** The instantaneous power is \( p_N(t) = A^2 \cos^2 \omega_0 t = A^2 \left(1 + \cos 2\omega_0 t\right)/2 \)

Average Power \( P_N = \langle p_N(t) \rangle = \frac{A^2}{T_0} \int_{-T_0/2}^{T_0/2} \left(1 + \cos 2\omega_0 t\right) dt = \frac{A^2}{2T_0} \left[ T_0, \frac{T_0}{2} \right] \)

Since \( \omega_0 = \frac{2\pi}{T_0} \), then \( \sin (2\omega_0 T_0/2) \) and \( \sin (-2\omega_0 T_0/2) \) are both equal to zero and

\[ P_N = \frac{A^2}{2T_0} \left[ \frac{T_0}{2} - \frac{T_0}{2} \right] = \frac{A^2}{2} \]

**Drill Problem 2.1 (Normalized Power)** - Enter normalized power for each signal in the column. The column total is provided as a checksum. A calculator should not be required for this problem.

<table>
<thead>
<tr>
<th>Signal</th>
<th>( P_N ) (watt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +10 \text{ V} ) ( -10 \text{ V} )</td>
<td>50</td>
</tr>
<tr>
<td>( 5 \text{ V} \cos 2\pi 50t )</td>
<td>12.5</td>
</tr>
<tr>
<td>( 12 \text{ V} \sin 2\pi 60t )</td>
<td>72</td>
</tr>
<tr>
<td>Checksum</td>
<td>143.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Signal</th>
<th>( P_N ) (watt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +3 \text{ V} ) ( -3 \text{ V} )</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Signal</th>
<th>( P_N ) (watt)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( +3 \text{ V} ) ( -3 \text{ V} )</td>
<td>2.25</td>
</tr>
</tbody>
</table>

Checksum → 63.25
2.2 Signal Gain Measurements

Signal gain is a measurement of an amplifying device or a transmission medium over which the signal propagates. The ratio of input and output power (or voltage) is usually expressed in logarithmic units and this is convenient for analyzing a series cascade of elements.

2.2.1 Power Gain in decibels

The ear responds to sound levels in an approximately logarithmic manner and thus acoustic power gain or loss ratios are expressed in a base 10 logarithmic units. A power ratio of 10 is defined as one Bel (to honor Alexander Graham Bell) but since human ears can distinguish sound power differences that are less than one Bel, the adopted unit is the decibel (dB). To determine the amplification of a system we apply an input signal, measure the input and output signal power and then express the dimensionless ratio in dB.

\[
\text{Power Gain (dB)} = 10 \log_{10} \left( \frac{P_o}{P_{in}} \right).
\]

Virtual Laboratory 2.1 - estimate the minimum difference in dB that your ears can detect. Consider an absolute measurement (for example, one day to the next) and a relative measurement (one second to the next).

Since power measurement with a traditional wattmeter is not practical for low power communication signals, measurements are made with a rms responding voltmeter. Power ratio is then calculated using circuit impedance values.

\[
G(\text{dB}) = 10 \log_{10} \left( \frac{V_o^2}{V_{in}^2} \right) \left( \frac{R_{in}}{R_L} \right).
\]

Example 2.2 - an amplifier has an input of 3 watts and output of 600 watts, what is the power gain?

Answer: Power gain (dB) = \(10 \log_{10} \left( \frac{P_o}{P_{in}} \right) = 10 \log_{10} (200) = 23 \text{ dB}\)

2.2.2 Voltage Gain

The decibel unit of power gain has been generalized to describe the voltage ratio in systems where the input and output impedances are not defined. For example, an ideal transformer with turns ratio 1:2 will have no power gain (0 dB) but will have a voltage gain of 2 (6 dB). Voltage gain is useful to describe electronic amplifiers which have high input impedance and low output impedance. In this case, voltage gain is constant but the power gain varies with the load resistance. As illustrated in the "rules of thumb" table, a 1 dB change
represents approximately 10% change in voltage. When the gain is less than unity, we refer to the device as having loss or attenuation and thus the dB values are not expressed as negative.

\[ \text{Voltage gain (dB)} = 10 \log_{10} \left( \frac{V_o}{V_{in}} \right) = 20 \log_{10} \left( \frac{V_o}{V_{in}} \right). \]

**Voltage gain vs frequency** - in audio systems, gain is usually specified at 1000 Hz and its variation with frequency is known as the **frequency response** or sometimes the amplitude response or magnitude response. In telephone systems, for example, gain is reduced at low frequencies by transformer coupling and reduced at high frequencies by transmission line loss as illustrated Figure 2-3. A more complete specification is given by the Transfer function \( H(f) \) which includes both amplitude (i.e gain) and phase angle.

**Phase vs frequency** response is a companion to Figure 2-3. An ideal system will have constant delay between input and output and is known as a linear phase system since the output phase increases in proportion to frequency.

**Example 2.3** - an amplifier has input of 2 mV rms and output of 400 mV rms. Determine the voltage gain and also the voltage gain expressed in dB.

**Answer:** Voltage Gain = 400mV / 2 mV = 200

\[ \text{Gain (dB)} = 20 \log_{10} 200 = 46.0 \text{ dB} \]

**Drill Problem 2.2 (dB)** – The gain properties of four amplifiers are listed in the table below. Complete the missing table entries and verify using the checksum.

<table>
<thead>
<tr>
<th>Voltage Gain</th>
<th>Amp #1</th>
<th>Amp #2</th>
<th>Amp #3</th>
<th>Amp #4</th>
<th>Checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power Gain</td>
<td>2</td>
<td>100</td>
<td>25</td>
<td>10,000</td>
<td>10,129</td>
</tr>
<tr>
<td>Gain (dB)</td>
<td>20</td>
<td>80</td>
<td></td>
<td></td>
<td>80</td>
</tr>
</tbody>
</table>

**2.2.3 Signal Distortion (Linear and non-linear)**

A transmission system or amplifier with inductive, capacitive and resistive elements will have variable gain and phase shift over the frequency range of interest. Mathematically, this is expressed as a transfer function, \( G(f) \). A signal, such as a pulse waveform, is distorted in this type of system and since the distortion is caused by linear circuit elements, we refer to this as “linear” distortion. The signal waveform can be restored by compensating this gain and delay (i.e. phase) distortion with a device known as an equalizer. In this type of signal distortion, no new frequencies are generated.
Non-linear distortion occurs in amplifiers that have reduced gain at higher amplitudes. This effect is known as saturation and a graph of output vs input voltage is not a straight line. This input-output amplitude non-linearity introduces “harmonic distortion” in single frequency signals and introduces “inter-modulation distortion” in signals with several frequency components (i.e. new frequencies are generated). Non-linear distortion is often correctable, but with somewhat more difficulty than for linear distortion.

### 2.3 SIGNAL LEVEL MEASUREMENTS

#### 2.3.1 dBm

The absolute power of a signal is commonly expressed in dB relative to one-milliwatt. Since wattmeters are not able to measure small powers, voltage-measuring instruments have been calibrated to read the power that would be delivered to a 50 Ω, 75 Ω or 600 Ω load. For example, a voltmeter reading of 0.775 Vrms may be calibrated to read 0 dBm which corresponds to 1 mW dissipation in a 600 Ω load. Function generators and meters calibrated for 50 Ω systems read 0 dBm for 0.225 Vrms signals. For larger signals, the calibration may be in dBW (i.e. the reference power is one watt). To properly measure complex waveforms, a dBm meter should respond in proportion to the “true rms” voltage.

\[
\text{Level (dBm)} = 10 \log_{10} \left( \frac{P_o}{1 \text{ mW}} \right)
\]

**Example 2.4** - The decibel unit can be used to express signal amplitude as compared to a reference power or voltage. The dBm unit has a one milliwatt reference and is normally used to quantify the level (or amplitude) of communication signals. Calculate the signal level (in dBm) of a 2 Vp-p sinusoid which is present at a 600 Ω load.

\[
P_{in} = \frac{V_{rms}^2}{R} = \frac{(1V/\sqrt{2})^2}{600\Omega} = 0.8333 \text{ mW} \quad P(\text{dBm}) = 10 \log_{10} \left( \frac{0.8333 \text{ mW}}{1 \text{ mW}} \right) = -0.79 \text{ dBm}
\]

**Example 2.5** - The rms voltage across the 300 Ω antenna input terminals of an FM receiver is 20 µV. Find the input power (watts), evaluate the input power as measured in decibels below 1 mW (dBm).

\[
P_{in} = \frac{(20 \mu V)^2}{300\Omega} = 1.333 \times 10^{-12} \text{ watts (1.33 pW)}
\]

\[
P(\text{dBm}) = 10 \log_{10} \left( \frac{1.33 \times 10^{-12} \text{ W}}{1 \text{ mW}} \right) = -88.7 \text{ dBm}
\]
2.3.2 dBV

This unit expresses voltage level as a ratio to one volt rms. No impedance level is specified (although 1000Ω impedance could be assumed to make dBm = dBV). Since this unit is clearly a voltage measurement, it is a preferred unit for systems with electronic amplifiers. While most radio frequency (RF) test equipment if for 50-ohm impedance, cable television (CATV) systems are based on 75-ohm interfaces, so to avoid confusion, the cable television industry uses the dBmV unit (1 mV reference). The dBV meter has a single calibration scale and there is no confusion about readings taken on 50Ω, 75Ω or 600Ω systems.

\[
\text{Level (dBV)} = 20\log_{10}\left(\frac{V_o}{1 \text{ Vrms}}\right).
\]

### Drill Problem 2.3 (dBW)

Observe the sample calculation and complete the tables to one decimal place accuracy. A calculator should not be required.

<table>
<thead>
<tr>
<th>Signal</th>
<th>P norm (watts)</th>
<th>Level (dBW 1Ω)</th>
<th>Signal</th>
<th>P norm (watts)</th>
<th>Level (dBW 1Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 volts rms</td>
<td>4</td>
<td></td>
<td>1 volt rms</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>20 volts rms</td>
<td>400</td>
<td></td>
<td>10 volts rms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 V p-p sinusoid</td>
<td>0.5</td>
<td>0.9 Vrms</td>
<td>2 Vrms</td>
<td>8</td>
<td>+9</td>
</tr>
<tr>
<td>4V cos 2 π 1000t</td>
<td>8</td>
<td>+9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Checksum</td>
<td>412.5</td>
<td>38</td>
<td>Checksum</td>
<td>105.8</td>
<td>+25</td>
</tr>
</tbody>
</table>

### Example 2.6 - dBm, dBW and dBV values depend on the measured voltage and the load impedance of the system being measured. Note the 30 dB difference between dBm and dBW

<table>
<thead>
<tr>
<th>Measured Voltage</th>
<th>dBV</th>
<th>dBm 600 Ω</th>
<th>dBm 75 Ω</th>
<th>dBm 50 Ω</th>
<th>dBW 50 Ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Vrms</td>
<td>0 dBV</td>
<td>2.2 dBm</td>
<td>11.8 dBm</td>
<td>13 dBm</td>
<td>-17 dBW</td>
</tr>
<tr>
<td>0.775 Vrms</td>
<td>-2.2 dBV</td>
<td>0 dBm</td>
<td>9 dBm</td>
<td>10.8 dBm</td>
<td>-19 dBW</td>
</tr>
</tbody>
</table>

2.3.3 Volume Unit (Vu) *

The volume unit was developed empirically to measure compound signals in a way that corresponds to the human ear. The purpose is to present consistent loudness to the listener and to avoid transmission system overload. It is used to continuously monitor transmission levels of voice and music. To model the response of the human ear, the meter has a 30-15,000 Hz frequency response and it responds with approximately 170 ms averaging time constant. Vu meters are incorporated in high-quality audio recorders.

The volume unit meter responds to rms voltage and is calibrated to read 0 Vu when a 0 dBm (0.775 Vrms at 600 Ω) sine wave is measured. The meter is calibrated in logarithmic units and the peak observed needle deflection is taken as the reading. The observation period is 60 seconds for music and 10 seconds for telephone conversations.
2.3.4 Cascaded Gains

In systems with cascaded elements, the dBm (or dBV) level of the output is easily calculated by adding all dB gains to the dBm (or dBV) level of the input. It is standard practice to express all levels, gains, signal to noise ratios etc. in logarithmic dB units.

2.3.5 Summation of Non-Coherent Tones

While Coherent tones with the same frequency and phase “add on a voltage basis”, sinusoids of the same frequency with phase difference of 90 degrees “add on a power basis”. In power addition, the resulting voltage amplitude is calculated as the square root of the sum of the component amplitudes squared. Same frequency signals with an arbitrary phase difference can be resolved into in-phase components and quadrature components. These are added separately on a voltage basis then the two resulting sums are combined using the square root of the sum of the squares (power addition).

Non-Coherent tones of different frequency may be considered as two signals with a steadily increasing phase difference. Since, over time, the phase difference is uniformly distributed in the range $0 \leq \theta < 2\pi$, the signals will add on a power basis. The following relations summarize the linear addition of sinusoidal signals.

- **coherent** $P_t = \frac{(V_{p1} + V_{p2})^2}{2}$ when $f_1 = f_2$ and $\theta_1 = \theta_2$

- **non-coherent** $P_t = P_1 + P_2$ when $f_1 \neq f_2$ or when $f_1 = f_2$ and $\theta_1 = \theta_2 \pm \pi/2$

Virtual Laboratory 2.2 - For two equal amplitude sinusoids of the same frequency, each with power $P = A^2/2$, in-phase addition results in $P = 2A^2$ (quadruple power), while in-phase subtraction results in $P = 0$. Addition at an arbitrary phase angle results in intermediate values of power. For uniform distribution of phase shift in the range $0 \leq \theta < 2\pi$, the average power is $P = A^2$. 

![Cascaded Gain Elements](image-url)
2.3.6 Summation of Signals that are Quantified in dB

When two signals of different frequency are summed, the resulting level is not simply the sum of the constituent dBm levels. Instead, the sum of the input signal powers must be used. The level $L_T$ of two added signals with dBm levels $L_1$ and $L_2$ is calculated by summing signal powers as follows:

$$L_T = 10 \log_{10} \left( \log^{-1} \left( \frac{L_1}{10} \right) + \log^{-1} \left( \frac{L_2}{10} \right) \right)$$

**Example 2.7** - Two tones at 900 and 1100 Hz are added to form a signal. The tones have levels, -6 dBm and -9 dBm respectively. Calculate the dBm level of the combined signal.

$$10 \log_{10} \left( \frac{P_{900}}{1 \text{ mW}} \right) = -6 \text{ dBm} \quad \Rightarrow \quad P_{900} = \log^{-1} \left( -0.6 \right) = 0.251 \text{ mW}$$

$$P_{1100} = \log^{-1} \left( -0.9 \right) = 0.126 \text{ mW}$$

$$P_{\text{total}} = 0.377 \text{ mW} \quad \Rightarrow \quad 10 \log_{10} \left( \frac{P_{\text{total}}}{1 \text{ mW}} \right) = -4.24 \text{ dBm}$$

**Example 2.8** - Two "in phase" tones at 1000 Hz are added and both tones have level -2 dBm. Calculate the dBm level at the combined signal.

$$P_1 = P_2 = \log^{-1} \left( -\frac{2}{10} \right) = 0.631 \text{ mW} \quad \text{and} \quad A_1 = A_2 = \sqrt{2} P_1 = 3.55 \times 10^{-2} \text{ V}$$

$$P_T = (A_1 + A_2)^2 / 2 = 2.52 \times 10^{-3} \text{ W} = +4.02 \text{ dBm} \quad \text{(i.e. } -2 \text{ dBm + "6 dB" = 4 dBm})$$

**Drill Problem 2.4 (dBm/dBV)** - Complete the following tables using one decimal place accuracy.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Level (dBm 600Ω)</th>
<th>Level (dBV)</th>
<th>P norm (Watts)</th>
<th>Signal</th>
<th>Level (dBV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.775 volts rms</td>
<td></td>
<td></td>
<td></td>
<td>4 volt rms</td>
<td></td>
</tr>
<tr>
<td>10 volts rms</td>
<td></td>
<td></td>
<td></td>
<td>4V cos2π100t + 3V cos2π100t</td>
<td></td>
</tr>
<tr>
<td>5 mV rms</td>
<td></td>
<td></td>
<td></td>
<td>4V cos2π100t + 3V cos2π120t</td>
<td></td>
</tr>
<tr>
<td>10 V p-p sinusoid</td>
<td></td>
<td></td>
<td></td>
<td>4V cos2π100t + 3V cos2π200t</td>
<td></td>
</tr>
<tr>
<td>Checksum</td>
<td>-8.4</td>
<td>-17.2</td>
<td></td>
<td>Checksum</td>
<td>65.5</td>
</tr>
<tr>
<td>Checksum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+47.9</td>
</tr>
</tbody>
</table>

Checksum = 8.4 - 17.2 = 65.5 + 47.9
2.4 MATHEMATICAL REPRESENTATION OF A SINUSOID

Sinusoids are the most fundamental of all signals and a thorough understanding of three forms of representation is essential for the study of communication systems.

1. Trigonometric Representation - sinusoidal signals are most simply represented in cosine form. A phase shift can be included as part of the argument.

\[ v(t) = A \cos(\omega_0 t) \]

or \[ v(t) = A \cos(\omega_0 t + \theta) \]

The polar form \( v(t) = A \cos(\omega_0 t + \theta) \) can also be represented in Cartesian form (with in-phase and quadrature components) as \( v(t) = x_1 \cos \omega_0 t - y_1 \sin \omega_0 t \) where \( x_1 = A \cos \theta \) and \( y_1 = A \sin \theta \). Conversion from Cartesian to polar form is accomplished using

\[ \theta = \tan^{-1}(y_1/x_1) \]

and

\[ A = \sqrt{x_1^2 + y_1^2} \].

2. Sum of complex exponentials - Sinusoidal signals may also be represented by sum of two complex exponentials. Similarly, a phase shifted waveform \( v(t) = A \cos(\omega_0 t + \theta) \) is broken into Cartesian cosine and sine terms, \( v(t) = x_1 \cos \omega_0 t - y_1 \sin \omega_0 t \), then expressed as the sum of complex exponentials using the Euler relations

\[
2 \cos \theta = e^{j \theta} + e^{-j \theta} \quad \text{and} \quad 2 \sin \theta = e^{j \theta} - e^{-j \theta}.
\]

When the sinusoid is expressed in trigonometric form, the normalized power has been shown to be \( A^2/2 \). When represented as the sum of two complex exponentials, each exponential has constant amplitude and power equal to \((A/2)^2\). The normalized power sum in the two exponentials is thus \((A/2)^2 + (A/2)^2 = A^2/2\).

3. Real part of a complex exponential - Sinusoidal signals may also be represented by the real part of the complex exponential (i.e. a rotating vector). The multiplying term \( Ae^{j \theta} \) is known as a phasor since it defines the phase and amplitude of \( v(t) \) with respect to the reference vector \( e^{j \theta} \).

\[ v(t) = \text{Re}\{Ae^{j \omega_0 t}\} = A \cos(\omega_0 t) \]

or \[ v(t) = \text{Re}\{Ae^{j \theta} e^{j \omega_0 t}\} = A \cos(\omega_0 t + \theta) \]
2.4.1 Complex Envelope Representation of a Modulated Sinusoid

A sinusoid that has time variation in both amplitude and phase can be represented as

\[ v(t) = \text{Re}\left\{ A(t)e^{j\theta(t)}e^{j\omega_c t} \right\} = A(t)\cos[\omega_c t + \theta(t)] . \]

The complex exponential is multiplied by a time varying phasor or “complex envelope” where \( A(t) \) represents amplitude (or “envelope”) variation and \( \theta(t) \) represents phase variation. We define the complex envelope in polar form as

\[ g(t) = A(t)e^{j\theta(t)} . \]

The time varying sinusoid \( v(t) = \text{Re}\left\{ g(t)e^{j\omega_c t} \right\} \) can also be represented in Cartesian form by expanding \( g(t) = x(t) + jy(t) \) and this results in

\[ v(t) = \text{Re}\left\{ x(t)e^{j\omega_c t} + jy(t)e^{j\omega_c t} \right\} = x(t)\cos \omega_c t - y(t)\sin \omega_c t . \]

**Virtual Laboratory 2.3 - Complex Exponential**

a) When viewed from the end of the positive time axis, does the vector \( e^{j\omega_c t} \) rotate clockwise or counter-clockwise?
b) The real (horizontal) portion of \( e^{j\omega_c t} \) represents sine or cosine?
c) Complex envelope – is it amplitude or phase modulation?

**Example 2.9 – Complex Envelope** - Representation of amplitude modulation using complex envelope (phasor) notation.

**Example 2.10 – Complex Envelope** – Representation of phase modulation using complex envelope notation.
2.5 **Frequency Spectrum Representation of Periodic Waveforms**

A periodic waveform is composed of repeated copies of a waveform segment where the segment length is equal to the repetition period. These waveforms are classed as power signals since the duration is assumed to be infinite and there is no decay. Examples of periodic waveforms are a sinusoid, a square wave and a triangular wave. A lengthy example might be a continuously repeating musical recording. We are particularly interested in pulse waveforms since they are used in sampling, an essential foundation of digital communication.

2.5.1 **Spectral Representation of a Sinusoid**

A sinusoid of the form \( v(t) = 100 \text{ mV} \cos 2\pi 2000t \) can be observed with an oscilloscope, an instrument that displays voltage versus time. The sinusoid can also be observed using a spectrum analyzer where the horizontal scale calibrated in hertz and vertical scale calibrated in Vrms, dBV or in dBm (50Ω). Figure 2-5 shows oscilloscope and spectrum analyzer displays of the sinusoid described above.

![Figure 2-5 Corresponding oscilloscope and spectrum analyzer displays](image)

This spectrum analyzer employs a narrow band-pass filter that is slowly swept over the frequency range of interest. The filter output voltage drives the vertical deflection of the screen while the horizontal deflection is proportional to filter frequency. The bandwidth of the filter is normally adjustable, for example 10 Hz, 30 Hz, 100 Hz and so on. Narrow bandwidth gives good spectral resolution but the sweep speed must be reduced in proportion to the square of the resolution bandwidth. Swept-frequency spectrum analyzers are commonplace at high frequencies, however, modern analyzers now use FFT processing for low frequency signals.

2.5.2 **Which Symbol for Frequency - \( \omega \) or \( f \)?**

Radio frequency allocations, equipment specifications, laboratory instruments and non-sinusoidal signals are described cycles per second or hertz. While we sometimes use radian frequency \( \omega \) (radians per second) to aid compactness, the variable \( f \) (hertz) will be used throughout our discussion of communication systems. The two frequency variables are simply related by \( \omega = 2\pi f \).
2.5.3 Repetitive Waveforms - Fourier Series Representation

A real periodic signal \( w(t) \) with period \( T_0 \) and frequency \( f_0 = 1 / T_0 \) can be represented as a series of sinusoids having frequencies at integer multiples (or harmonics) of the fundamental frequency \( f_0 \). This is known as a Fourier series and the sinusoids, when added together, produce a signal equivalent to the original signal \( w(t) \). The zero\(^{th} \) harmonic represents a dc component, the average value of the waveform. In Cartesian trigonometric form, the Fourier series consists of cosine and sine terms which are orthogonal functions (quadrature terms separated by 90°). In polar trigonometric form we have a series of cosine terms, each with magnitude and phase shift. Thus we have

\[
w(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \quad \text{(Cartesian trigonometric form)}
\]

and

\[
w(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n \omega_0 t + \theta_n) \quad \text{(polar trigonometric form)}
\]

where the coefficients are related as \( D_0 = a_0 \), \( D_n^2 = a_n^2 + b_n^2 \) and \( \theta_n = \tan^{-1} \left( -b_n / a_n \right) \). Plots of these coefficients versus frequency are the magnitude spectrum and phase spectrum.

Using Euler’s identity, we have an equivalent polar exponential form and an equivalent Cartesian exponential form where coefficients are complex. If we set coefficients

\[
c_0 = a_0, \quad c_n = \frac{1}{2} \left( a_n - j b_n \right) \quad \text{and} \quad c_{-n} = \frac{1}{2} \left( a_n + j b_n \right)
\]

then

\[
\sum_{n=1}^{\infty} c_n e^{j n \omega_0 t} = \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \quad \text{since} \quad e^{j \omega t} = \cos \omega t + j \sin \omega t \quad \text{and}
\]

\[
\sum_{n=1}^{\infty} c_{-n} e^{-j n \omega_0 t} = \frac{1}{2} \sum_{n=1}^{\infty} a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t) \quad \text{since} \quad e^{-j \omega t} = \cos \omega t - j \sin \omega t .
\]

We thus have

\[
w(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{j n \omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-j n \omega_0 t} \quad \text{(Cartesian exponential form)}
\]

and

\[
w(t) = \sum_{n=-\infty}^{\infty} c_n e^{j n \omega_0 t} \quad \text{(compact Cartesian exponential form)}
\]

We verify by substituting this compact expression into the calculation of the \( c_n \) coefficients,

\[
c_n = \frac{1}{T_0} \int_{T_0}^{a+T_0} w(t) e^{-j n \omega_0 t} \, dt \quad \text{and, since} \quad \int_{T_0}^{a+T_0} e^{j n \omega_0 t} \, dt = 0 \quad \text{when} \quad n \neq m, \quad \text{this simply results in}
\]

\[
c_n = \frac{1}{T_0} \int_{a}^{a+T_0} c_n e^{j n \omega_0 t} e^{-j n \omega_0 t} \, dt = c_n .
\]

If \( w(t) \) is real and even (symmetric about \( t = 0 \)) then \( c_n \) is only real, but if \( w(t) \) is real and odd (inverse symmetric about \( t = 0 \)), then \( c_n \) is only imaginary. To clarify, we number quadrants in a counter-clockwise direction about the origin starting with the top right as
quadrant #1. In even symmetry, the waveform is flipped about the vertical axis, quadrants 1 and 2 are mirror images across the \( t = 0 \) axis and so are quadrants 3 and 4. In odd symmetry, the waveform is flipped around the vertical axis then flipped around the horizontal axis.

**Example 2.11: Symmetry** - illustration of even and odd symmetric functions. Even functions are symmetric across the boundary between quadrants 1 and 2. Odd functions are inverse symmetric across the boundary.

<table>
<thead>
<tr>
<th>Quadrants</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

### 2.5.4 Repetitive Pulse Waveforms - Time Series Representation

A repetitive waveform can be expressed as the sum of delayed versions of a waveform segment. As a simple example, we consider a single rectangular pulse waveform \( p(t) \). The single pulse is mathematically described by \( \text{rect}(t/\tau) \) or by \( \Pi(t/\tau) \) (this character has been chosen to resemble the time waveform). The expression for a periodic rectangular pulse sequence, \( w(t) \), is given below.

\[
w(t) = \sum_{k=-\infty}^{\infty} \Pi(t-kT_0/\tau)
\]

Figure 2-6 Expression for a periodic pulse waveform

Sampled signals form the basis for modern communication systems thus we are interested in their analysis. A convenient model is multiplication of the input signal by a periodic pulse signal with magnitudes 0 and 1. To aid the analysis of sampled signals, we study time and frequency representations of periodic (repetitive) pulse waveforms.

### 2.5.5 Repetitive Pulse Waveform - Fourier Series Representation

Consider a repetitive pulse waveform of amplitude \( A \) with period \( T_0 = 5 \text{ ms} \) and pulse width \( \tau = 1 \text{ ms} \) (duty cycle = 0.2). The waveform, \( w(t) \), can be expressed as the sum of Fourier series components as

\[
w(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}
\]

where \( \omega_0 = 2\pi f_0 = 2\pi/T_0 \). This waveform is real and thus \( c_n = c_{-n} \). Also we select the \( t = 0 \) axis to give an even function (symmetric about \( t = 0 \)) and thus all \( c_n \) are real (no imaginary portion). Since the waveform is repetitive and of infinite duration, we have a “line spectrum” composed of the repetition frequency and integer multiples of that frequency. For the rectangular pulse waveform, we calculate \( c_n \) as
Chapter 2: Representation of Signals

For the specific rectangular waveform with \( A = 1 \) V, \( T = 5 \) ms and \( \tau = 1 \) ms, the Fourier coefficients are illustrated as spectral lines versus frequency. Vertical units are in volts (peak not rms).

Two-sided Amplitude Spectrum

exponential components

\[ w(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 t} \]

One-sided Amplitude Spectrum

sinusoidal components

\[ w(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n) \]

**Example 2.12 – Pulse Spectrum:** Calculate the first eight Fourier series coefficients for unit amplitude pulse sequences with duty cycles \( 3/4, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7 \) and \( 1/8 \).

**Answer:**

<table>
<thead>
<tr>
<th></th>
<th>( c_0 )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>( c_3 )</th>
<th>( c_4 )</th>
<th>( c_5 )</th>
<th>( c_6 )</th>
<th>( c_7 )</th>
<th>( c_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>duty = ( 3/4 )</td>
<td>0.750</td>
<td>0.225</td>
<td>-0.159</td>
<td>0.075</td>
<td>0</td>
<td>-0.045</td>
<td>0.053</td>
<td>-0.032</td>
<td>0</td>
</tr>
<tr>
<td>duty = ( 1/2 )</td>
<td>0.500</td>
<td>0.318</td>
<td>0</td>
<td>-0.106</td>
<td>0</td>
<td>0.064</td>
<td>0</td>
<td>-0.045</td>
<td>0</td>
</tr>
<tr>
<td>duty = ( 1/3 )</td>
<td>0.333</td>
<td>0.276</td>
<td>0.138</td>
<td>0</td>
<td>-0.069</td>
<td>-0.055</td>
<td>0</td>
<td>0.039</td>
<td>0.034</td>
</tr>
<tr>
<td>duty = ( 1/4 )</td>
<td>0.250</td>
<td>0.225</td>
<td>0.159</td>
<td>0.075</td>
<td>0</td>
<td>-0.045</td>
<td>-0.053</td>
<td>-0.032</td>
<td>0</td>
</tr>
<tr>
<td>duty = ( 1/5 )</td>
<td>0.200</td>
<td>0.187</td>
<td>0.151</td>
<td>0.101</td>
<td>0.047</td>
<td>0</td>
<td>-0.031</td>
<td>-0.043</td>
<td>-0.038</td>
</tr>
<tr>
<td>duty = ( 1/6 )</td>
<td>0.167</td>
<td>0.159</td>
<td>0.138</td>
<td>0.106</td>
<td>0.069</td>
<td>0.032</td>
<td>0</td>
<td>-0.023</td>
<td>-0.034</td>
</tr>
<tr>
<td>duty = ( 1/7 )</td>
<td>0.143</td>
<td>0.138</td>
<td>0.124</td>
<td>0.103</td>
<td>0.078</td>
<td>0.050</td>
<td>0.023</td>
<td>0</td>
<td>-0.017</td>
</tr>
<tr>
<td>duty = ( 1/8 )</td>
<td>0.125</td>
<td>0.122</td>
<td>0.113</td>
<td>0.098</td>
<td>0.080</td>
<td>0.059</td>
<td>0.038</td>
<td>0.017</td>
<td>0</td>
</tr>
</tbody>
</table>
Virtual Laboratory 2.4 - Observe the square wave approximation that has been generated using a multi-sinusoid waveform synthesizer.

a) The desired square waveform has amplitude two-volt peak-to-peak and zero average voltage. What is the normalized power?

b) What are the theoretical peak-to-peak voltages of the first three components? (checksum = 3.905)

c) What is the normalized power sum of the first three synthesizer components? (checksum = 0.933)

Drill Problem 2.5 (Fourier Series) - Determine the peak, peak-to-peak, average and rms voltages of the following two signals x(t) and y(t). Also determine the normalized power, fundamental frequency, period and the first six coefficients of the complex "exponential" Fourier series. Complete the table to two decimal places.

\[ x(t) = 7.07V \cos 2\pi 50t \]

| | \( V_p \) | \( V_{p-p} \) | \( V_{rms} \) | \( |f| \) | \( T_0 \) | \( C+\delta \) | \( C+\gamma \) | \( C+\tau \) | \( C+t \) | \( \text{Check-} \) |
|---|---|---|---|---|---|---|---|---|---|---|
|x(t)| 7.07 | 0.00 | 25.00 | 0.15 | 10 ms | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 124.75 |
|y(t)| 2.00 | | | | | | | | | | 69.83 |

2.5.6 Sinc Function and Sampling Function

The term \( \sin(\pi f t)/\pi f t \) appears frequently in communication theory and, to simplify notation, many textbooks have defined \( \text{sinc}(x) = \sin(\pi x)/\pi x \) while others have defined the sampling function \( \text{Sa}(\pi x) = \sin(\pi x)/\pi x \). Not all books are consistent and, to avoid possible confusion, these notations have not been applied in this workbook. The expression \( \sin(\pi f \tau)/\pi f \tau \) is written in full. The sinc function diminishes with time and represents an energy waveform.

2.5.7 Periodic Signals - Spectral Power and Parseval’s Theorem

A periodic signal, \( w(t) \), is a power waveform and may be represented as a series of component signals at frequencies harmonically related to the fundamental frequency. For these signal components, the power in the total signal is the sum of the powers associated with each component as given by

\[
P_f = \sum_{n=-\infty}^{+\infty} |c_n|^2 \quad \text{(a sum to infinity)}.
\]

Parseval’s theorem states that this calculation of signal power in the frequency domain should equal \( P_t \), the time average of instantaneous power. Thus for periodic signals

\[
P = \frac{1}{T_o} \int_{a}^{a+T_o} |w(t)|^2 \, dt = \sum_{n=-\infty}^{+\infty} |c_n|^2 \quad \text{(i.e. } P = P_t = P_f)\]
Example 2.13 – Parseval’s Theorem  Illustrate the time waveform of a ±2 V squarewave with frequency 2 kHz. Calculate the normalized power in the waveform. Also illustrate Fourier series coefficients on a 2 sided frequency scale for frequencies in the range ±15 kHz. Calculate the power sum of the frequency components up to 15 kHz. Do your results show \( P_f \approx P_t \) as consistent with Parseval’s theorem?

Example spectral component

\[
c_5 = 4V\left(1\right)\left(\sin\left(\frac{\pi}{2}\right)\left(\frac{250}{500}\right)\right) = 0.254 V
\]

Sum of power spectral components

\[
P_f = (0.0)^2 + 2\left[(1.273)^2 + (-0.424)^2\right] + (0.254)^2 + (-0.182)^2 = 3.796 Watts
\]

2.5.8 Repetitive Pulse Waveform – Spectral Effects of \( \tau \) and \( T \)

Variation in pulse width and pulse period change the pulse waveform’s spectral components as illustrated in Figure 2-9. In the initial spectrum, components occur at 200, 400, 600 Hz …. , (multiples of \( 1/T_o \)). Spectral nulls occur at 1 kHz, 2 kHz, 3 kHz …. , (multiples of \( 1/\tau \)).

Figure 2-9 Fourier coefficients of a rectangular pulse rain when \( T \) and \( \tau \) are varied.
2.6 Frequency Spectrum Characterization of Non-repetitive Signals

2.6.1 Energy Signals – Fourier Transform and Rayleigh’s Energy Theorem*

We now consider “one-shot” energy signals that have insignificant amplitude after some finite duration. First, we assume the waveform repeats at interval $T_0$ and thus we initially obtain a Fourier series representation

$$ w(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nf_0 t} = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} w(\tau) e^{-j2\pi nf_0 \tau} d\tau \right] e^{j2\pi nf_0 t} . $$

To analyze the single waveform, we let $T_0$ approach infinity so $f_0$ (i.e. $1/T_0$) becomes the differential frequency $df$, and $nf_0$ becomes the continuous variable $f$. We replace summation with integration and the calculation of $w(t)$ becomes the inverse Fourier transform integral

$$ w(t) = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} w(\tau) e^{-j2\pi f\tau} d\tau \right] e^{j2\pi ft} df . $$

The portion in square brackets is defined as amplitude spectral density $W(f)$. While Fourier series coefficients $c_n$ have units of volts, $W(f)$ has units of volt-seconds or volts/hertz (since the calculation does not include the factor $1/T_0$). $W(f)$ is two-sided with frequency ranging over $\pm \infty$. Calculation of amplitude spectral density $W(f)$ from the time waveform is known as the Fourier transform while the reverse process is the inverse Fourier transform.

$$ W(f) = \Im \{ w(t) \} = \int_{-\infty}^{\infty} w(t) e^{-j2\pi ft} dt \quad \text{(Fourier Transform)} $$

$$ w(t) = \Im^{-1} \{ W(f) \} = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} df . \quad \text{(Inverse Fourier Transform)} $$

Table 2-1 Fourier Transform Pairs.

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(f)$</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
<td>$e^{-\pi t^2}$</td>
<td>$e^{-\pi f^2}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\delta(f)$</td>
<td>$x'(t)$</td>
<td>$j2\pi fX(f)$</td>
</tr>
<tr>
<td>$\delta(t-t_0)$</td>
<td>$e^{-j2\pi ft_0}$</td>
<td>$\text{sgn}(t)$</td>
<td>$\frac{1}{j\pi f}$</td>
</tr>
<tr>
<td>$e^{-j2\pi ft_0}$</td>
<td>$\delta(f-f_0)$</td>
<td>$u(t)$</td>
<td>$1 \delta(f) + \frac{1}{j2\pi f}$</td>
</tr>
<tr>
<td>$\Pi(t/t)$</td>
<td>$\tau \text{sinc}(\pi f)$</td>
<td>$\sum_{n=-\infty}^{\infty} \delta(t-nT_0)$</td>
<td>$\frac{1}{T_0} \sum_{m=-\infty}^{\infty} \delta(f-mT_0)$</td>
</tr>
<tr>
<td>$f_b \text{sinc}(f_b \cdot t)$</td>
<td>$\Pi(f/f_b)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Also of interest is the (normalized) **energy spectral density** calculated as \( E(f) = |W(f)|^2 \) with units of **joules/hertz**. Total energy in an energy waveform may be calculated as the integral of energy spectral density over the entire frequency range. **Rayleigh’s energy theorem** (similar to Parseval's theorem) equates the energy calculated in the time domain with that calculated from the signal's Fourier transform.

\[
E = \int_{-\infty}^{\infty} |w(t)|^2 \, dt = \int_{-\infty}^{\infty} |W(f)|^2 \, df
\]

**Example 2.14 – Fourier Transform of a Rectangular Pulse** - Calculate the amplitude spectral density \( W(f) \) and the total spectral energy of a single 3 volt, 5 s pulse. Verify Rayleigh’s (Parseval’s) theorem by calculating time domain energy.

\[
w(t) = A \Pi \left( \frac{t}{\tau} \right) \quad \text{where } t = 5 \text{ s and } A = 3 \text{ volts}
\]

\[
W(f) = \frac{A}{-j2\pi f} \left[ e^{-j2\pi ft/\tau} - e^{+j2\pi ft/\tau} \right]_{-\tau/2}^{\tau/2} = A \left[ \frac{\sin \pi tf}{\pi tf} \right]_{-\tau/2}^{\tau/2} = 15[\sin(5\pi f)/5\pi f] \text{ volts/Hz}
\]

<table>
<thead>
<tr>
<th>Time Domain Energy</th>
<th>Spectral Energy</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3V)^2 \cdot 5 \text{ s} = 45 \text{ joules} )</td>
<td>(= 2A^2 \tau^2 \int_{-\infty}^{\infty} \frac{\sin \pi tf}{\pi tf}^2 , df )</td>
<td>(= \frac{2A^2 \tau^2}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} = 45 \text{ J} )</td>
</tr>
</tbody>
</table>

**Example 2.15 – Fourier Transform of a Sinc Pulse** - A sinc waveform in the time domain and a rectangular spectrum form a Fourier transform pair. Calculate the Fourier transform of \( w(t) = 12V \sin(3\pi t)/3\pi t \) and show that the normalized energy in the waveform is 48 Joules.

**Answer:**

We know the Fourier transform of a rectangular pulse of width \( \tau \) results in a sinc \( \pi tf/\pi tf \) shaped spectrum. Thus we know the Fourier transform of the pulse \( A \sin(\pi tf)/\pi tf \) will be a rectangular spectrum with width \( f_0 \) and amplitude \( A/f_0 \). In this case the rectangular spectrum is:

\[
W(f) = A \Pi \left( \frac{f}{3 \text{ Hz}} \right) \text{ volts/Hz}
\]

\[
\int_{-\infty}^{\infty} |W(f)|^2 \, df = \int_{-1.5}^{1.5} |W(f)|^2 \, df = 48 \text{ joules}
\]

\( W(f) \) is a complex function of frequency which can be written in Cartesian form as \( W(f) = X(f) + jY(f) \) or in polar form as \( W(f) = |W(f)| e^{j\phi(f)} \). The symmetry properties of \( W(f) \) in the Fourier Transform are similar to those of \( c_n \) in the Fourier series.

If \( w(t) \) is real, \( W(f) = W^*(-f) \) i.e. \( \phi(f) = -\phi(-f) \)

If \( w(t) \) is real and even, then \( W(f) \) has only a real part.

If \( w(t) \) is real and odd, then \( W(f) \) has only an imaginary part.
In summary, we consider \( w(t) \) to be a real voltage waveform that might be displayed on an oscilloscope. The voltage fluctuates with time and thus there are associated frequency (spectral) components. The Fourier transform gives an amplitude spectrum showing the strength of one frequency component relative to others.

### 2.6.2 Noise – a continuous non-repetitive power signal

Communication systems are impaired by distortion, interference and noise. While distortion can often be reversed and interference somewhat mitigated by filters, noise remains as a system limitation. To succeed in battle against noise, we must understand the enemy. Noise, in many cases, is an added random voltage with Gaussian amplitude distribution and uniform (“white”) power spectral density (PSD) over the frequency range of interest. For example, *Johnson noise* results from thermal agitation and has white spectral density.

Virtual Laboratory 2.5 – Observe the waveform, spectrum, and sound of the sum of 16 sinusoids in the frequency range 2500 – 3000 Hz. Compare with white noise bandlimited to 2500 – 3000 Hz.

Pop up a trace of the two waveforms. Can you see any significant difference in the signal waveforms?

White noise can be viewed as an infinitely large number of equal-amplitude sinusoids separated by an infinitely small frequency increment. Fourier transform yields a continuous frequency distribution. To allow application of the Fourier transform, the aperiodic semi-infinite random noise waveform, \( w(t) \), is truncated to form the energy waveform, \( w_E(t) \).

Application of the Fourier transform yields the amplitude spectrum \( W_E(f) \) in volts/hertz and the energy spectral density \( |W_E(f)|^2 \) in joules/hertz. Dividing by \( T \), yields the power spectral density, \( S_W(f) \), in units of watts/hertz.

\[
S_W(f) = \lim_{T \to \infty} \frac{1}{T} |W_E(f)|^2 \quad \text{power spectral density}
\]

For the range of frequencies used in communication systems, thermal noise has uniform spectral density, \( S_N(f) = N_0 \), so *noise power is proportional to transmission bandwidth*. In double-sided spectra, power spectral density is denoted \( N_0 / 2 \).

Thermal noise is present in all electronic components including resistors. Thermally excited electrons in the conducting medium undergo collisions resulting in an abrupt change in current flow. If these collisions are modeled as impulses, the energy spectral density from each impulse is “flat” (uniform spectral density vs frequency) and, since the ensemble of impulses occur randomly, the power spectrum of the total noise is also flat.

**Amplitude spectral density** of noise is computed as the square root of \( S_W(f) \) and has units of \( \text{volts/Hz}^{1/2} \). As an example, the TL081 operational amplifier is specified to have \( 17 \text{nV/Hz}^{1/2} \) equivalent input noise at frequencies near 1 kHz.

In a swept-frequency spectrum analyzer, a narrow band-pass filter is slowly swept over the frequency range of interest and the filter output voltage drives the vertical deflection of the screen. If the filter bandwidth is increased, more frequency components are included and the output noise voltage increases in proportion to the square root of bandwidth. When measuring audio signals, the filter bandwidth (a.k.a. resolution bandwidth) might be 30 Hz.
Modern spectrum analyzers use the discrete Fourier transform (DFT) or the fast Fourier transform (FFT) to compute the spectral display of a sampled segment of the signal. This segment is repeated to form a periodic signal so the resulting transform is discrete as in the Fourier series.

Virtual Laboratory 2.6 - measurement of lowpass filtered noise. Filters do not have sharp cutoff so measurements deviate may slightly from theory.

- a) Determine the meter reading in volts when using 10 kHz bandwidth. _____
- b) Estimate the meter reading in volts when using 25 kHz bandwidth. _____
- c) The display amplitude would reduce by what factor if resolution bandwidth is reduced from 100 Hz to 1 Hz? CHECKSUM 11.18

Virtual Laboratory 2.7 - measurement of bandpass filtered noise.

- a) Find the normalized noise power spectral density (in mW/Hz) _____
- b) Estimate the meter reading in volts for 5 - 15 kHz bandwidth. _____
- c) What is the noise level in dBm (600 ohms) when the meter reads 2.2 Vrms for 6 – 14 kHz bandwidth? CHECKSUM 13.7

Drill Problem 2.6 (Spectral Density) - Voltage and normalized power of three signals are measured directly and then measured after passing through 50 Hz and 80 Hz ideal lowpass filters. Complete the table.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Voltage after LPF</th>
<th>Power after LPF</th>
<th>Checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 50$ ms, $\tau = 25$ ms</td>
<td>$10 \cos 2\pi 60 t$</td>
<td>$No/2 = 20$ mW/Hz</td>
<td>Checksum</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P_n$ (W)</th>
<th>$V_{LPF\ 50}$ (V rms)</th>
<th>$V_{LPF\ 80}$ (V rms)</th>
<th>$P_{LPF\ 50}$ (W)</th>
<th>$P_{LPF\ 80}$ (W)</th>
<th>Checksum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+10 \text{ V}$</td>
<td>$-10 \text{ V}$</td>
<td>$+10 \text{ V}$</td>
<td>$-10 \text{ V}$</td>
<td>$+10 \text{ V}$</td>
<td>$-10 \text{ V}$</td>
</tr>
<tr>
<td>$1.41$</td>
<td>$50$</td>
<td>$10.41$</td>
<td>$83.06$</td>
<td>$18.35$</td>
<td>$143.26$</td>
</tr>
</tbody>
</table>
2.7 **Signal Amplitude Characterization**

To overcome the effects of noise, it is desirable to transmit the largest possible signal power. However, to avoid signal clipping, peak voltages must be constrained within the system voltage limits. This tradeoff is especially problematic in OFDM multicarrier systems such as digital subscriber line (DSL) systems and wireless computer access to the Internet.

2.7.1 **Peak to rms ratio (Crest Factor)**

When compared to a sinusoid, information signals such as voice, music and DSL voltages have much larger peak values. Thus, when the peak voltage is constrained (by amplifier supply voltage for example), information signals will have lower rms voltage and lower signal power. Peak to rms ratio (or crest factor) gives a rough characterization of amplitude variation. A more complete characterization is given by the amplitude probability density function (pdf).

2.7.2 **Histogram and probability density function (PDF) of a sinusoid**

Amplitude probability density function (PDF) is introduced by considering the amplitude histogram of the signal. Assume that the voltage range of a signal is divided into a number of voltage “bins” and a histogram is calculated to show the probability of the signal voltage being within the range of each voltage bin. As the number of bins is increased and the range of each bin approaches zero, the probability of bin occupancy approaches zero. However, by dividing the probability of bin occupancy by the bin range, we obtain the amplitude probability density function expressed in units of probability/volt.
In the illustration Figure 2-11, we consider a 20-bin histogram of a unit amplitude sinusoid. The sinusoid “takes a long time to turn around” at the peaks and therefore spends a large fraction of each cycle in the uppermost and lowest voltage bins. If the bin width is reduced and the number increased, the shape of the histogram will approach that of the probability density. Probability density is proportional to the reciprocal of the waveform slope and approaches infinity at the waveform peaks.

2.7.3 Sum of independent sinusoids, convolution and Gaussian distribution

Numerical convolution is now illustrated with an example. Using a computer, two unit amplitude sinusoids were added and the amplitude histogram was calculated using 0.1 volt bins over the voltage range ±2 V. The sinusoid frequencies were 1000 Hz and 1220 Hz. The empirical histogram below has a maximum around zero volts with the probability of bin occupancy diminishing at higher voltages.

As an alternative to empirical measurement, we use numerical convolution to calculate the combined signal histogram from the component histograms of \( x(t) \) and \( y(t) \). In the following demonstration, we consider the specific cases that result in an output of 1.8 volts. The three required voltage sums are 0.8+1.0, 0.9+0.9, and 1.0+0.8. As illustrated in Figure 2-13, we “flip” one of the component histograms so as to align pairs of bins that sum to 1.8 volts. If sinusoids \( x(t) \) and \( y(t) \) are independent, the joint probabilities for these three events are 0.048(0.141), 0.062(0.062) and 0.141(0.048) respectively and the output 1.8 volts occurs with probability 0.0174. Probability for an output of 1.7 volts is found by “sliding” the “flipped” histogram so that 4 pairs of bins align with each pair summing to 1.7 volts. Calculation of the complete output histogram by this method is known as discrete convolution of the component histograms which is expressed as

\[
P_{z=m} = \sum_{n=-0.95}^{n=0.95} P_{x=n}P_{y=m-n}.
\]

As bin size is reduced, we approach continuous convolution

\[
P_{z}(v = V) = \int_{v-1}^{v+1} P_{x}(v)P_{y}(V-v)dv = P_{x}(v)*P_{y}(v).
\]
2.7.4 Noise – Amplitude Characterization

When a large number of independent signals are summed, the resulting PDF approaches the “bell shaped” curve of a Gaussian density function. This property is formally discussed in the central limit theorem.

This Gaussian amplitude density function is important in the study of communication systems since many types of noise (particularly thermal noise) have approximately Gaussian amplitude densities. From the above study of two added sinusoids, it can be appreciated that the sum of many equal amplitude sinusoids (as in “white” noise), would approximate a Gaussian amplitude density.

Virtual Laboratory 2.8 - see how amplitude probability density (PDF) can be indirectly observed on a spectrum analyzer through the use of a voltage to frequency converter (note that the spectrum analyzer normally shows amplitude spectral density). Observe the probability density when 2, 4 and 8 sinusoids are added and investigate the bell-shaped amplitude spectral density for bandlimited noise.
2.8 MEAN, STANDARD DEVIATION, CORRELATION AND CONVOLUTION

2.8.1 Signal mean

The mean of a signal (its dc value) is given by the time average of the signal or by the Fourier component $W(0)$. The time average is computed over a finite observation interval as

$$m_w = W(0) = \langle w(t) \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} w(t) \, dt$$

For periodic waveforms, $t_2 - t_1$ is set equal to the period and for non-periodic waveforms, the average is obtained as $t_2 - t_1$ approaches infinity.

2.8.2 Variance and Standard Deviation

The variance $\sigma^2$ of a signal is a measure of the variation about its mean. The standard deviation $\sigma$ is simply the square root of the variance. For voltage signals with zero mean, the variance equals the signal power and the standard deviation is equal to the root-mean-square (rms) value.

$$\sigma_w^2 = \langle (w(t) - m_w)^2 \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (w(t) - m_w)^2 \, dt$$

Drill Problem 2.7 (Amplitude Characterization) – Consider the sum of four equal amplitude “raised cosine” voltages of the form $x(t) = 10 \, \textrm{V} + 10 \, \textrm{V} \, \cos 2\pi f_n t$ where the four frequencies are unrelated. For the total waveform, compute the following:

Mean voltage $m = \_\_\_\_\_\_\_\_\_ \, \textrm{V}$, Variance $\sigma^2 = \_\_\_\_\_\_\_\_\_ \, \textrm{V}^2$, standard Deviation $\sigma = \_\_\_\_\_\_\_\_\_ \, \textrm{V}$

Normalized Power = ___1800_W, rms voltage = ___________V

Checksum = 2096.57

2.8.3 Correlation coefficient

The correlation coefficient indicates a linear relationship between signals $x(t)$ and $y(t)$. When $x(t)$ changes, is there a corresponding change in $y(t)$? It tells us, on average, how well a straight line can be fit to the sample values of $x(t)$ and $y(t)$. For continuous signals the correlation coefficient is given by

$$\rho = \frac{\mu_{x,y}}{\sigma_x \sigma_y} = \frac{\langle (x(t) - m_x)(y(t) - m_y) \rangle}{\sqrt{\langle (x(t) - m_x)^2 \rangle \langle (y(t) - m_y)^2 \rangle}}$$

where $m_x$ and $m_y$ are the mean values of $x(t)$ and $y(t)$. Including $\sigma_x, \sigma_y$ in the denominator normalizes the expression so $\rho$ falls in the range $-1 \leq \rho \leq 1$. A similar expression can be written for sampled data.

$$\rho = \frac{\sum_i (x_i - m_x)(y_i - m_y)}{\sqrt{\sum_i (x_i - m_x)^2} \sqrt{\sum_i (y_i - m_y)^2}}$$
If an x-y display of \( x(t) \) and \( y(t) \) is a line with positive slope extending from the first to the third quadrant, the correlation coefficient is \( \rho = 1.0 \). If the x-y display is a line with negative slope then the correlation coefficient is \( \rho = -1.0 \). A circular x-y display of the two variables indicates correlation coefficient \( \rho = 0.0 \).

**Example 2.16** – Calculate the correlation coefficient of the first four digits in the numbers \( \pi \) and \( e \).

\[
\pi = 3.14159 \quad m_x = \frac{1}{N} \sum_{i=1}^{N} x_i = \frac{1}{4} (3+1+4+1) = 2.25
\]

\[
e = 2.71828 \quad m_e = \frac{1}{N} \sum_{i=1}^{N} x_i = \frac{1}{4} (2+7+1+8) = 4.5
\]

\[
\rho = \frac{(3-2.25)(2-4.5) + (1-2.25)(7-4.5) + (4-2.25)(1-4.5) + (1-2.25)(8-4.5)}{\sqrt{(3-2.25)^2 + (1-2.25)^2 + (4-2.25)^2 + (1-2.25)^2} \cdot \sqrt{(2-4.5)^2 + (7-4.5)^2 + (1-4.5)^2 + (8-4.5)^2}}
\]

\[
\rho = \frac{-1.875 - 3.125 - 6.125 - 4.375}{\sqrt{6.75} \cdot \sqrt{37}} = -0.98
\]

**Virtual Laboratory 2.9** - Correlation of audio signals can be seen on an oscilloscope and can also be observed using stereo headphones.

a) Observe and listen to X-Y noise. Does the “fuzzy ball” correspond to two correlated or uncorrelated noise sources?

b) Observe and listen to X-Y stereo music. In which case does the sound seem to be in the middle of your head?

**2.8.4 Cross-correlation**

Radar, sonar, and medical ultrasound use correlation to determine the magnitude and delay of a returned echo signal and thus estimate the size and distance of a reflecting object. Since the echo signal \( y(t) \) is displaced in time from the transmitted signal \( x(t) \), the simple correlation coefficient will not suffice. To address this problem, we introduce the variable time-shift \( \tau \) to advance the echo signal and then compute correlation as a function of this time-shift. Cross-correlation is thus defined for power signals as

\[
R_{xy}(\tau) = \left\langle x^*(t)y(t+\tau) \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{t_1}^{t_1+T} x^*(t)y(t+\tau)dt
\]

While the correlation coefficient is a single value, the cross-correlation function is a function of relative delay. Also note that mean values are not eliminated from the calculation of \( R_{xy}(\tau) \) and there is no normalization.
2.8.5 Autocorrelation

The autocorrelation function $R_{xx}(\tau)$ measures the similarity between a signal $x(t)$ and a time-advanced copy of the same signal $x(t + \tau)$. The two signals match perfectly when $\tau = 0$ and, at that point, autocorrelation is equal to signal power. If the signal contains echoes, some degree of matching will occur at other values of $\tau$. Time variation of a physical waveform is thus evaluated by comparing the signal with various delayed versions of itself. For a periodic signal, autocorrelation will reach a maximum for time differences that are an integer multiple of the period. For example, autocorrelation of outside temperature will show maximums at 24-hour intervals. Since a signal and its time-shifted version are identical, we have symmetry about $\tau = 0$ and thus $R_{xx}(\#) = R_{xx}(\#)$. If the signal has consistent (stationary) mean and variance, $R_{xx}(\tau)$ will be independent of starting time $t_1$. For a power signal, the autocorrelation function is defined as:

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{t_1}^{t_1+T} x(t)x(t + \tau)dt$$

(power signal - watts).

And for the energy signal $x_T(t)$ (subscript $T$ indicates truncation to the interval $T$), the autocorrelation function is:

$$R_{E_{xx}}(\tau) = \int_{-\infty}^{\infty} x_T(t)x_T(t + \tau)dt$$

(energy signal - joules).

Example 2.18 A ship-to-ship communication system transmits repeated radio frequency (RF) bursts. For a single burst, with rectangular amplitude envelope, $x(t)$, the distant receiver receives two bursts, one from the direct point-to-point path and the second from a reflection off the water. The received envelope is described by $y(t)$. Observe the cross-correlation $R_{xy}(\tau)$ and the auto-correlation $R_{yy}(\tau)$.

2.8.6 Convolution

In the discussion of probability density of summed signals in Section 2.7.3 we introduced the concept of convolution. In subsequent chapters, convolution will be used to calculate the output spectrum when two or more signals are multiplied together. Similarly, for cascaded elements, where the system frequency response is the product of the individual element frequency responses, the time response is calculated through convolution of individual impulse responses. Convolution $g(\tau)$ of two energy signals is calculated as a function of displacement $\tau$ between signal $x_T(t)$ and a time-reversed version of the other signal, $h_T(t)$.

$$g(\tau) = x_T(t) \otimes h_T(t) = \int_{-\infty}^{\infty} x_T(t)h_T(\tau - t)dt$$
Taking the Fourier transform of both sides shows that multiplication in the frequency domain is equivalent to convolution in the time domain. It can be similarly shown that multiplication in the time domain is equivalent to convolution in the frequency domain.

\[
\mathcal{F}\{x_T(t) \otimes h_T(t)\} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x_T(t) h_T(\tau - t) \, dt \right] e^{-j\omega \tau} \, d\tau
\]

\[
\mathcal{F}\{x_T(t) \otimes h_T(t)\} = \int_{-\infty}^{\infty} x_T(t) \left[ \int_{-\infty}^{\infty} h_T(\tau - t) e^{-j\omega (\tau - t)} \, d\tau \right] e^{-j\omega t} \, dt
\]

\[
\mathcal{F}\{x_T(t) \otimes h_T(t)\} = \int_{-\infty}^{\infty} x_T(t) H(f) e^{-j\omega t} \, dt
\]

\[
\mathcal{F}\{x_T(t) \otimes h_T(t)\} = X(f)H(f)
\]

2.8.7 Autocorrelation and power spectral density*

To obtain an estimate of the power spectral density \( S_N(f) \) of a random noise signal, we select a finite-length sample \( n_T(t) \) of the waveform. While we could calculate the Fourier transform of this finite duration “energy” signal, a mathematical expression of the waveform segment is too complex for analytical purposes. However, the autocorrelation of the waveform segment is essentially quite short and very suitable for Fourier transform which yields the energy spectrum of the waveform segment. Dividing by the segment length \( T \), provides an estimate of the noise power spectral density. We begin with the previous expression for convolution

\[
\mathcal{F}\left\{ \int_{-\infty}^{\infty} x_T(t) h_T(\tau - t) \, dt \right\} = X(f)H(f)
\]

and replace \( x_T(t) \) with \( n_T(t) \) and also replace \( h_T(\tau - t) \) with a delayed version of the noise sample waveform \( n_T(t-\tau) \). Because of the time reversal in this latter replacement, we then replace \( H(f) \) with \( N^*(f) \). (Note that magnitude spectrum is independent of time reversal).

\[
\mathcal{F}\left\{ \int_{-\infty}^{\infty} n_T(t) n_T(t-\tau) \, dt \right\} = N(f)N^*(f)
\]

We recognize the right side as the energy spectral density \( E_n(f) \) and, by recalling \( R_{xx}(\tau) = R_{xx}(-\tau) \), we recognize the left side as the Fourier transform of signal autocorrelation.

\[
\mathcal{F}\left\{ \int_{-\infty}^{\infty} n_T(t) n_T(t-\tau) \, dt \right\} = E_n(f)
\]

This is a simplified case of the Wiener-Khintchine theorem. To estimate power spectral density \( S_n(f) \) we calculate time average (expected value) by dividing both sides by \( T \) and let \( T \to \infty \).

\[
\mathcal{F}\{R_{nn}(\tau)\} = S_n(f)
\]
2.9 **Chapter Summary**

*Practical signals* are real-valued functions that can be displayed on an oscilloscope. We define signals to be single valued functions of time. For mathematical convenience, we sometimes consider signals to be complex valued. Signals may be classified into basic categories: energy or power, periodic or non-periodic, deterministic or random.

*System outputs* occur in response to applied inputs and, if the system is linear, superposition applies and no new frequencies are generated. We say that the system is time-invariant, for example, if the input-output response is consistent at different times of day. System gains and signal amplitudes are expressed in dB for convenience. A series connection of several elements is analyzed by simply adding the dB values (instead of a lengthy multiplication). On the other hand, for the summation of signals with quadrature phase or different frequencies, the resulting power and rms voltage must be computed by adding the powers of the constituent signals.

An arbitrary waveform can be represented as a summation of sinusoids (our most basic waveform) and these sinusoids can be represented in trigonometric form or in complex exponential form. *Periodic waveforms* are represented by a Fourier series of sinusoids with frequencies at integer multiples of the fundamental (periodic) frequency. Plots of the sinusoid amplitude (in volts) and phase coefficients (in radians) versus frequency are called the amplitude spectrum and the phase spectrum.

To analyze the spectral content of *non-repetitive signals*, we adapt the Fourier series calculation and this results in the Fourier Transform. For energy signals (one-shot signals), we do not divide by the period $T$ so the amplitude spectrum has units of volt-seconds or volts/hertz; the energy spectrum has units of joules/hertz. Since the waveform is not repetitive, the spectrum is continuous (i.e. no discrete lines).

For *continuous non-repetitive* (random) signals, we truncate the waveform to the interval $T$, calculate the energy spectral density then divide by $T$ to get the power spectral density in units of watts/hertz. Amplitude spectral density is calculated by taking the square root of the power spectral density and has units of volts per root hertz.

In amplifiers and A/D converters, to improve the ratio of signal to noise, we try to maximize signal power and yet we need to avoid peak clipping. We therefore characterize signals in terms of *peak to rms ratio* to quantify the signal power for a given peak voltage. Full amplitude characterization is given by the probability density function (PDF) which takes on a Gaussian distribution when a large number of independent signals are summed together. We also characterize signals by similarity (*correlation*) to another signal or to delayed versions of itself.
Problems

p2.1 Can an energy signal be periodic? Yes No
The following expressions describe energy or power waveforms. Circle the power waveforms.

\[ 5V - 10V \sin(t), \quad 10V \delta(t), \quad 5V \cos(1000t), \quad 5V \sin(t) e^{-10t}, \quad 5V e^{-20t}, \quad 5V \sin(10t)/10t \]

p2.2 A test signal is specified as 10V cos 377 t.
   a) What is the normalized power of this signal?
   b) If the signal exists for exactly 30 sinusoidal cycles, determine the normalized energy.

p2.3 Two sinusoids are added. Calculate the average normalized total power and rms voltage when the
first sinusoid is 8V \cos(2\pi 1000t) and a) the second sinusoid is 6V \cos(2\pi 1000t + \pi/2), b) the second
sinusoid is 6V \cos(2\pi 1000t + \pi/4) and c) the second sinusoid is 6V \cos(2\pi 1200t).

p2.4 A signal is measured at 4.0 Vrms in a 50 ohm radio frequency cable. What is the signal level in
dBm and in dBW?

p2.5 An amateur radio signal traveling on a 50 ohm co-axial cable has sinusoidal waveform with peak to
peak voltage 7.07 volts. What is the signal level in dBm and dBV?

p2.6 An average reading power meter reads 3.5 W when connected at the output of a CATV amplifier.
The amplifier is driving a 75 ohm resistive load.
   a) What is the power in dBW?
   b) What is the power in dBm?
   c) What is the level in dBV?

p2.7 Several things are wrong with the following statement. “A system with gain 20 dBm has input level
+5 dBm and output level -15 dB”. Make the corresponding correct statement.

p2.8 Calculate the following to one decimal place.
   a) an audio amplifier has voltage gain of 400. What is the voltage gain in dB?
   b) a microwave amplifier has power gain of +13 dB. What is the ratio of output power to input
      power?

p2.9 A signal is composed of three linearly added tones at 200 Hz, 500 Hz and 1000 Hz. All three
components have level of -50 dBm. What is the average power of the combined signal?

p2.10 Three signals of different frequencies have levels -3, -5, and -10 dBm. What is the resulting level if
the three signals are combined? Note that signals of different frequencies add on a power basis.

p2.11 A speech signal with average power 0 dBm is added to a 60 Hz tone with level -6 dBm. What is the
total resultant power expressed in mW?

p2.12 A +20 dBm signal is added to a +30 dBm signal of different frequency. What is the sum in dBm?

p2.13 A DVD player outputs a -17 dBm, channel 3 signal on a 75 ohm cable while a video cassette player
outputs a -12 dBm, channel 4 signal. Both signals are linearly added without loss to form signal X
on a 75 ohm cable. Determine the rms voltage levels of signals 3, 4 and X.

p2.14 The decibel unit can be used to express signal amplitude as compared to a reference power or
voltage. The dBm unit has a one milliwatt reference and is normally used to quantify the level (or
amplitude) of communication signals.
   a) A 2V p-p sinusoid is present at a 600 ohm load. Calculate the signal level in dBm.
   b) Two tones at 900 and 1100 Hz are added to form a signal. The tones have levels, -6 dBm and -9
dBm respectively. Calculate the dBm level of the combined two tone signal.
c) Two "in phase" tones at 1000 Hz are added. Both tones have level -2 dBm. Calculate the dBm level of the two-tone signal.

p2.15 A telephone station set generates two tones at 941 Hz and 1204 Hz when the "*" button is pressed. The expression
\[ v(t) = 0.8 \cos(2\pi \times 941t) + 1.0 \cos(2\pi \times 1204t) \]
describes the generated voltage (in volts) and the waveform is illustrated below. Calculate the following:

a) The normalized average power (in watts) of: i) the 941 Hz signal component, ii) the 1204 Hz signal component and iii) the total combined signal.
b) The actual power (in dBm) of: i) the average combined signal and ii) the instantaneous peak power of the combined signal. Assume that the telephone line impedance is 600 ohms.
c) The “beat” frequency (i.e. the envelope variation frequency).

p2.16 When viewed from the end of the positive time axis, which way does the vector \( e^{it} \) rotate? Is \( e^{it} \) analogous to a “right hand thread” or a “left hand thread.” Is this the same thread as the screw cap on a pop bottle?

p2.17 A modulated signal can be expressed in complex envelope form as
\[ v(t) = \text{Re}\{g(t) e^{j\omega t}\} \]
where the complex envelope \( g(t) = 4 + 3 \cos 377t \) and the signal frequency \( \omega = 2000\pi \). Sketch the first 20 ms of this waveform.

p2.18 Four rectangular waveforms \( s_W, s_X, s_Y \) and \( s_Z \) are shown; a) indicate the frequency on each waveform, b) what is the Fourier coefficient \( c_3 \) for the waveform \( s_Y \) and c) which waveform has coefficients: \( c_0 = 0.250, c_1 = c_{-1} = 0.225, c_2 = c_{-2} = 0.159 \).

Ch2-Sig-12g.doc
In the illustration below, the signal \( x(t) \) is a 60 Hz, 12 volt (rms) cosine wave. This signal is passed through an ideal full wave rectifier and then a highpass filter that removes the dc component and frequencies up to 2 Hz. Sketch the waveforms at points A, B, and C. Determine the normalized power of signals \( x(t) \), \( |x(t)| \) and \( y(t) \). Hint: signals of different frequency add (subtract) on a power basis; this includes signals at zero frequency (dc).

\[
x(t) = 17V \cos(2\pi 60t)
\]

Using calibrated axes, sketch the time waveform of \( w(t) = \sin(2t) = \sin(2\pi t)/2\pi t \).

A single rectangular pulse in the time domain and a sinc function in the frequency domain are a Fourier transform pair as illustrated in Example 2.15. Similarly, a rectangular spectrum and a sinc function in the time domain form a Fourier transform pair as illustrated in Example 2.16. Calculate the Fourier transform of the waveform \( w(t) = 6V \sin(3\pi t)/3\pi t \). Show that the normalized energy in the waveform is 12 joules.

The signal \( m(t) \) with Fourier transform \( M(f) \) enters a signal processing device which has the output \( M_x(f) = M(f)e^{j5f} \). What specific function is performed by the signal processor?

Using calibrated axes, sketch the spectrum of the time waveform \( w(t) = \sin(4\pi t)/4\pi t \).

A noise-like signal, \( v(t) \), is composed of sinusoidal components each with amplitude 1 Vrms and spaced at one-tenth hertz intervals. Each component has normalized power equal to one watt. The one-sided spectrum is illustrated below.

\[
v(t) = \sum_{n=0}^{n=+\infty} \sqrt{2} \cos(0.2\pi nt)
\]

a) What is the power spectral density (watts/Hz) and the amplitude spectral density (volts/√Hz) of the noise-like signal?
b) The noise-like signal passes through a filter with bandwidth two hertz. What is the normalized power and the rms voltage of the filter output signal?
c) Sketch the amplitude density versus voltage (i.e. PDF) for the filter output and indicate the standard deviation (in volts).

Draw lines between each time waveform and the appropriate frequency domain representation(s).

i) Fourier series (V)  
ii) Energy spectral density (J/Hz)  
iii) Amplitude spectral density (V/Hz)  
iv) Power spectral density (W/Hz)  
v) Amplitude spectral density (V/√Hz)

A rms responding meter reads 1.5 volts when measuring a white noise source that is bandlimited by a 20 kHz lowpass filter. Determine the meter reading when the filter cutoff is reduced to 5 kHz.

The (normalized) single sided power spectral density, \( S(f) \), of a telephone modem signal is illustrated. Calculate normalized power and the level in dBV for the signal.

The noise in an audio system has amplitude spectral density 0.045 V/√Hz.  
am) calculate the rms noise voltage is the noise is bandlimited to 10 kHz.
b) determine the two-sided power spectral density (PSD), $N_0/2$.

p2.30b A FM receiver is tuned to a distant station and the audio output has excessive white noise. The noise is measured at 100 mVrms ($P_n = 10$ mW). Using the tone control, the output bandwidth is reduced from 10 kHz to 2 kHz. What are the new values of noise voltage and noise power?

p2.31 The input to the system shown below is zero mean, Gaussian white noise $n(t)$ with normalized two-sided power spectral density ($N_0/2$) equal to 5 mW/Hz. The system is a 10 Hz ideal lowpass filter. Determine the normalized power at the output $y(t)$, sketch the amplitude probability density (i.e. probability density vs voltage) and determine the mean and standard deviation. Hint: the standard deviation is equal to the rms voltage.

\[ n(t) \xrightarrow{10 \text{ Hz LPF}} y(t) \]

p2.33 The differentiator and 10 Hz ideal lowpass filter illustrated below have equivalence to noise processing in a FM receiver. The input is a zero mean, Gaussian white noise $n(t)$ with normalized two-sided power spectral density ($N_0/2$) equal to 5 mW/Hz. Determine the normalized power at the output $y(t)$. At each frequency, the differentiator has voltage gain $2\pi f$ and power gain $(2\pi f)^2$.

\[ n(t) \xrightarrow{\frac{\text{d}}{\text{dt}}} x(t) \xrightarrow{10 \text{ Hz LPF}} y(t) \]

p2.34 Peak to RMS ratio is a convenient way to determine maximum signal power when the peak-to-peak voltage is limited by amplifier or A/D voltage limits. Determine, from first principles, the peak to rms ratio of a triangular waveform with zero mean. (Integration is required)

p2.35 The following 3 waveforms have zero average value and peak to peak voltage of 20 volts. Given the peak-to-rms ratio (crest factor), calculate the normalized power in each case.

<table>
<thead>
<tr>
<th>Waveform</th>
<th>Peak/rms</th>
<th>Norm. Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square wave</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Sine wave</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>Triangular wave</td>
<td>$\sqrt{3}$</td>
<td>$\sqrt{3}$</td>
</tr>
</tbody>
</table>

p2.36 A periodic waveform $w(t)$ consists of four 1 ms segments each with one cycle of sinewave. The sinewave amplitudes in the 4 successive segments are 2v, 4v, 2v, and 1v. The periodic waveform has a repetition rate of 250 Hz and has no discontinuities. Determine the average normalized power $<x^2(t)>$ and peak to rms ratio of the waveform.

p2.37 Determine the normalized power and peak to rms ratio of the waveform below.
p2.38 Consider a triangular waveform with zero average and peak voltages of ± 0.5 volts.
   a) Illustrate the probability histogram assuming 0.1 volt bins.
   b) Assume the addition of two such waveforms with identical amplitude but unrelated frequency.
      Illustrate the histogram of the combined signal.
   c) Roughly sketch the amplitude PDF of four added waveforms (recall the central limit theorem).

p2.39 Consider a speaker with the specifications approximately reproduced below.
   i) Assuming 80 watts peak instantaneous power, what average power might be expected for a music signal
      with peak to rms ratio of 4.0. ii) The specifications state the frequency range as 45 Hz to 22,000 Hz. What is
      your estimation of the ±3 dB frequency range of the speaker?

p2.40 A wavelength division multiplexed optical transmission system has 64 unmodulated optical carriers
   with 100 GHz spacing starting at 193 THz. Each carrier has rms amplitude 0.707Ep and peak amplitude Ep.
   Assuming that, at some point in time, all carriers simultaneously reach their peak value, what is the
   peak/rms ratio of the combined optical signal.

p2.41 Modern communication systems such as DSL Internet access and IEEE 802.11b wireless computer
   modems use multicarrier OFDM (orthogonal frequency division multiplexing). In these systems it
   is advantageous to phase the precisely spaced carriers to minimize the crest factor (peak/rms ratio). To
   demonstrate the effect of phasing, consider the transmitted signal
   \[ s(t) = \cos(2\pi 300t + \theta_3) + \cos(2\pi 400t + \theta_4) + \cos(2\pi 500t + \theta_5) \]
   and use a computer to evaluate the crest factor when a) \( \theta_3 = 0, \theta_4 = 0, \text{ and } \theta_5 = 0 \) and b) \( \theta_3 = \pi / 2, \theta_4 = 0, \text{ and } \theta_5 = \pi / 2 \)

p2.42 Calculate the correlation coefficient of the digits in the numbers 365 and 352.

p2.43 For continuous signals, an expression for the correlation coefficient is shown. Given the signals
   illustrated below, sketch \( x(t) - m_x \) and \( y(t) - m_y \), and their product and then calculate the correlation
   coefficient.

\[ \rho = \frac{\mu_{xy}}{\sigma_x \sigma_y} = \frac{\langle (x(t) - m_x)(y(t) - m_y) \rangle}{\sqrt{\langle (x(t) - m_x)^2 \rangle} \sqrt{\langle (y(t) - m_y)^2 \rangle}} \]
Chapter 2: Representation of Signals

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P2.43b A service technician is investigating the source of excessive noise in a FM stereo receiver. The right and left channel outputs are connected to an X-Y oscilloscope to display correlation of the two noisy outputs. There is no received signal. Select the oscilloscope display for each case below.

a) noise originating only in the left channel amplifier. Circle one: d), e), f)
b) noise in the receiver power supply, affecting both channels equally. Circle one: d), e), f)
c) noise originating in the L-R amplifier (see FM Stereo, Chapter 3). Circle one: d), e), f)

P2.44a Random binary information is transmitted by driving a laser with a series of pulses at the rate of 500 MHz. Logic 1 is represented by a 4-volt pulse and logic 0 by a 2-volt pulse. Between the pulses, the voltage is 1 volt. Sketch the autocorrelation of this information waveform over the range -7 ns < τ < 7 ns. What are the units for the vertical axis.

P2.44b A half wave cosine wave with amplitude 4 volts and frequency 60 Hz is centered about the \( t = 0 \) axis. Calculate the first \( c_0 \)-\( c_4 \) Fourier series coefficients of the half wave signal by using convolution of a cosine wave spectrum with that of a unit amplitude square gating waveform.

P2.45 Two rectangular pulse signals with levels 0 and +1 as illustrated are convolved to produce a pulse waveform with peak value \( +2.0 \times 10^{-6} \). Illustrate the resulting waveform.

P2.47 In the display of frequency spectra, several different units are used for the vertical axis depending on the characteristics of the time waveform. Possible units are volts, volts per hertz, volts per root hertz, watts, joules per hertz and watts per hertz.

a) What are the vertical axis units of the amplitude spectrum of a time function consisting of a single pulse, not repeated?
b) What are units for the amplitude spectrum if the time function is a periodically repeating pulse?
c) What are the units for the amplitude spectrum if the time function is a repeating pulse with random amplitude.
Virtual Laboratory 2.4 - Observe the square wave approximation that has been generated using a multi-sinusoid waveform synthesizer. a) The desired square waveform has amplitude two-volt peak-to-peak and zero average voltage. What is the normalized power? b) What are the theoretical peak-to-peak voltages of the first three components? (checksum = 3.905) c) What is the normalized power of the first three components summed and what rms voltage is expected? (0.933 W and 0.966 V).

Virtual Laboratory 2.6 - Access the virtual experiment on lowpass filtered noise. Observe the images of noise bandwidth and measured noise voltage. a) determine the normalized power spectral density, \( N_0 \), of the noise source (in mW/Hz) b) estimate the meter reading in volts and in dBm when the bandwidth is increased to 25 kHz.

Design Problem – (the first 7 pages of Chapter 3 will aid the solution of this problem) A traveling rock music group wants a portable instrument to analyze the echo acoustics of each venue prior to their concert. Using a random noise source \( x(t) \) on stage and a microphone \( y(t) \) at various places in the audience, results are to be recorded on a laptop computer with a 4-channel data acquisition capability. a) Prepare a block diagram of your design that includes an electronically controlled delay element, a 4-quadrant multiplier, and a sliding window averaging filter. b) Sketch the predicted test results if the microphone is located 100 m from the stage and there is a 20 ms delay in the first (and only) echo. Design criteria are as follows:

i) simultaneous measurement of cross-correlation and autocorrelation for delays up to 400 ms in steps of 2 ms. Bandwidth of signal processing 20 Hz to 10 kHz.

ii) Averaging filter (LPF) to minimize variation in the output due to double frequency products and 60 Hz interference.

iii) Measurement time less than 60 seconds. You will need to consider consider that the response time of the LPF is approximately the reciprocal of its bandwidth.

Extra Problems

Find the mean value (time average) and the power for the following signals:

(a) \( v(t) = 3V + 3V \cos(4000\pi t) \),

(b) \( v(t) = 3V \sin(40\pi t) + 4V \cos(4000\pi t) \)

Sketch the waveform and find the energy of the following signals:

(a) \( g(t) = \begin{cases} 
\sin(t) & \text{for } 0 < t \leq 2\pi \\
0 & \text{otherwise}
\end{cases} \)

(b) \( v(t) = \begin{cases} 
3V \sin(4\pi t) + 4V \cos(8\pi t) & \text{for } 0 < t \leq 0.5 \\
0 & \text{otherwise}
\end{cases} \)

Using calibrated axes, sketch the spectrum of the time waveform \( w(t) = \frac{\sin(4\pi t)}{4\pi} \).