Exponential Distribution

Most widely used probability distribution in reliability assessment.

Requirements:
- Events must be random
- Hazard rate must be constant

\[ R(t) = e^{-\lambda t} \quad Q(t) = 1 - e^{-\lambda t} \]

Mean or Expected value of \( f(x) \)
\[ E(x) = \int_{-\infty}^{\infty} x f(x) \, dx \]

Mean Time to Failure, \( MTTF = \int_{0}^{\infty} f(t) \, dt = \frac{1}{\lambda} \)

But, \( f(t) = \frac{dQ(t)}{dt} = \frac{d(1-e^{-\lambda t})}{dt} = \lambda e^{-\lambda t} \)

\[ MTTF = \int_{0}^{\infty} dR(t) = \int_{0}^{\infty} R(t) \, dt \]
Example: Exponential Distribution

Find the mean time to failure of a component which has a failure rate of 2 failures per year. Calculate its reliability for different mission times, e.g. 10, 1000, 10000 hours.

\[ \text{MTTF} = \frac{1}{\lambda} = \frac{1}{2} = 0.5 \text{ yrs} = 0.5 \times 8760 \text{ hrs} \]

\[ R(t) = e^{-\lambda t} \]

\[ R(10) = 0.997719, \quad R(1000) = 0.795877, \quad R(10000) = 0.101967 \]

![Graph showing reliability over time](image)

Failure Probability in a Time Interval

A priori probability of failure in time interval \( t \), \( Q(t) = 1 - e^{-\lambda t} \)

A Priori Probability:
probability calculated by logically examining existing information

A Posteriori Probability:
conditional probability that is assigned after relevant information is taken into account.

The probability of failure in the next interval \( t \) actually depends conditionally upon its behavior preceding that interval.
e.g. it cannot fail in that interval if it already failed prior to that interval

It is, therefore, required to determine (a posteriori) probability of a component failing in an interval \( t \) given that it has survived prior to that interval.
A Posteriori Probability

Probability of component failing during \( t \) given that it has survived up to \( T \), \( Q_c(t) \)

\[
P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\int_T^{t+T} f(t)dt}{\int_T^T f(t)dt}
\]

But, \( f(t) = \lambda e^{-\lambda t} \)

Event
- A: failure during \( t \) (shaded area)
- B: surviving up to \( T \) (colored area)

\[
Q_c(t) = \frac{\int_T^{t+T} \lambda e^{-\lambda t} dt}{\int_T^T \lambda e^{-\lambda t} dt} = \frac{e^{-\lambda T} - e^{-\lambda(T+t)}}{e^{-\lambda T}} = 1 - e^{-\lambda t} = \text{a priori probability } Q(t)
\]

Reliability evaluation in the useful life of a component is, therefore, relatively simple as exponential distribution is applicable.

In the wear-out phase, conditional probability must be used.

Exponential Distribution Applications

**Series Systems**

\[
R_s = R_1 R_2 = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2)t}
\]

\[
R_s(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\sum_{i=1}^{n} \lambda_i t} = e^{-\lambda t}
\]

i.e. resulting distribution for the system is also exponential

**Parallel Systems**

\[
Q_s = Q_1Q_2 = \prod_{i=1}^{n} [1 - e^{-\lambda_i t}]
\]

\[
R_s(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-\sum_{i=1}^{n} \lambda_i t} = e^{-\lambda t}
\]

Product rule of unreliability. \( Q_s = Q_1 \cdot Q_2 \)

\[
R_s = R_1 + R_2 - R_1R_2
\]

\[
R_s = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}
\]

cannot obtain equivalent hazard rate for exponential distribution

resulting distribution for the system is non-exponential

resulting hazard rate for the system is no longer constant, but a function of time
Partially Redundant (m out of n) Systems

Apply Binomial Expansion

\[ [R(t) + Q(t)]^n = \sum_{i=0}^{n} \binom{n}{r} R(t)^{n-r} Q(t)^r \]

During useful life period when component failures are exponentially distributed

\[ R(t) = e^{-\lambda t} \quad \text{and} \quad Q(t) = 1 - e^{-\lambda t} \]

Example:

A simple electronic circuit consists of 6 transistors each having a failure rate of 10^{-6} f/hr, 4 diodes each having a failure rate of 0.5 \times 10^{-6} f/hr, 3 capacitors each having a failure rate of 0.2 \times 10^{-6} f/hr, 10 resistors each having a failure rate of 5 \times 10^{-6} f/hr and 2 switches each having a failure rate of 2 \times 10^{-6} f/hr. Assuming connectors and wiring are 100% reliable, evaluate the equivalent failure rate of the system and the probability of the system surviving 1000 and 10000 hours if all components must operate for system success.

Equivalent failure rate of the system, \( \lambda_e = \sum_{i=1}^{n} \lambda_i \)

\[ = 6(10^{-6}) + 4(0.5 \times 10^{-6}) + 3(0.2 \times 10^{-6}) + 10(5 \times 10^{-6}) + 2(2 \times 10^{-6}) = 6.26 \times 10^{-5} \text{ f/hr} \]

\[ R_s(1000 \text{ hr}) = e^{-\lambda_e t} = e^{-6.26 \times 10^{-5} \times 1000} = 0.9393 \]

\[ R_s(10,000 \text{ hr}) = e^{-\lambda_e t} = e^{-6.26 \times 10^{-5} \times 10000} = 0.5347 \]
Example:
Consider a system comprising of 4 identical units each having a failure rate of 0.1 f/yr. Evaluate the probability of the system surviving 0.5 years and 5 years if at least two units must operate successfully.

Using Binomial Expansion,
\[ [R(t) + Q(t)]^4 = R^4(t) + 4 R^3(t)Q(t) + 6 R^2(t)Q^2(t) + 4 R(t)Q^3(t) + Q^4(t) \]

where, \( R(t) = e^{-\lambda t} \) and \( Q(t) = 1 - e^{-\lambda t} \)

For 2 out of 4 system,
\[ R_s(t) = R^4(t) + 4 R^3(t)Q(t) + 6 R^2(t)Q^2(t) \]
\[ = e^{-4\lambda t} + 4 e^{-3\lambda t} (1 - e^{-\lambda t}) + 6 e^{-2\lambda t} (1 - e^{-\lambda t})^2 \]

For \( t = 0.5 \) years, \( \lambda t = 0.1 \times 0.5 = 0.05 \) \( \rightarrow R_s(0.5) = 0.9996 \)

For \( t = 5 \) years, \( \lambda t = 0.1 \times 5 = 0.5 \) \( \rightarrow R_s(5) = 0.8282 \)