Orthogonal Frequency-Division Multiplexing (OFDM)\footnote{This note is largely based on Chapter 12 of textbook “Wireless Communications” (Cambridge University Press, 2005), authored by Prof. Andrea Goldsmith, Department of Electrical Engineering, Stanford University).}

1 Introduction

We have learned that QAM is a combination of amplitude modulation (ASK) and phase modulation (PSK). On the other hand, FSK uses multiple orthogonal carrier frequencies to send information bits. Specifically, for FSK modulation with $N$ frequencies, only one of $N$ frequencies is activated over one symbol duration of $T_s = \lambda T_b$, where $T_b$ is the bit duration. What frequency that is activated over one symbol duration is determined by the mapping from $\lambda$ information bits to the frequency value (or index).

OFDM is essentially a combination of QAM and FSK. Instead of activating only one out of $N$ carriers as in FSK, in OFDM all the carriers can be activated all the time and QAM symbols are sent over each carrier. This is illustrated in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Spectrum of OFDM signals.}
\end{figure}

Referring to the spectrum of OFDM signals, one has the following observations:

- The data rate on each of the subchannels is much less than the total data rate, and the corresponding subchannel bandwidth is much less than the total system bandwidth.

- The number of subchannels can be chosen so that each subchannel has a bandwidth small enough so that the frequency response over each subchannel’s frequency range is approximately constant. This ensures that inter-symbol interference (ISI) on each subchannel is small.
The subchannels in OFDM need not be contiguous, so a large continuous block of spectrum is not needed for high rate transmission.

- The modulation formats on different subchannels need not be the same. In fact one may adaptively choose different modulation schemes according to the instantaneous quality of the subchannels.

The most attractive feature of OFDM is that its modulator and demodulator can be efficiently implemented with DSP. As will be seen later, with the DSP implementation, the ISI can be completely eliminated through the use of a cyclic prefix. The main technical issues that impair performance of OFDM are frequency offset and timing jitter, which degrade the orthogonality of the subchannels. In addition, having QAM signals transmitted simultaneously over all carriers, the peak-to-average power ratio (PAPR) of the composite OFDM transmitted signals is significantly higher than that of QAM signals transmitted on a single carrier. This is a serious problem when nonlinear amplifiers are used.

OFDM is currently used in many wired and wireless systems. However, it is not a new technique. It was first used for military HF radios in the late 1950’s and early 1960’s. Starting around 1990, OFDM has been used in many diverse wired and wireless applications, including digital audio and video broadcasting in Europe, digital subscriber lines (DSL), and the most recent generation of wireless LANs. There are also a number of newly emerging applications of OFDM, including fixed wireless broadband services, mobile wireless broadband known as FLASH-OFDM, and data-over-cables systems (DOCSIS 3.1). OFDM has also been adopted as the air interface in the 4th-generation cellular systems (e.g., LTE-Advanced).
2 OFDM Viewed as Multicarrier Modulation with Overlapping Subchannels

Figure 2 shows how the OFDM signal is obtained as a sum of $N$ QAM signals, one is centered at its own carrier frequency. Let $T_N$ be the duration of each OFDM symbol. Then the relationship between $T_N$ and the information bit rate $r_b$ (or the bit duration $T_b = 1/r_b$) can be found as follows. Let $M_n = 2^{\lambda_n}$ be the size of the QAM constellation used on carrier $f_n$, which means that each QAM symbol sent over the $n$th channel can carry $\lambda_n$ bits. The total number of bits sent over all channels in each OFDM symbol is therefore equal to

$$\lambda = \sum_{n=0}^{N-1} \lambda_n.$$  

It follows that $T_N = \lambda T_b = \frac{\sum_{n=0}^{N-1} \lambda_n}{r_b}$.

Let $X_n[k] = V_{I,n}[k] + jV_{Q,n}[k]$ represent the QAM symbol that is transmitted on the $n$th carrier during the $k$th OFDM symbol. It then follows that the transmitted signal corresponding to the $k$th OFDM symbol (which carries a total of $\lambda$ bits) is given as:

$$s(t) = \sum_{n=0}^{N-1} (V_{I,n}[k]p(t) \cos(2\pi f_n t + \phi_n) - V_{Q,n}[k]p(t) \sin(2\pi f_n t + \phi_n))$$

$$= \sum_{n=0}^{N-1} \Re \{ X_n[k]p(t)e^{j(2\pi f_n t + \phi_n)} \}, \quad (k-1)T_N \leq t \leq kT_N \quad (1)$$

In the above expression, $p(t)$ is the baseband shaping filter and $\phi_n$ is the phase of the $n$th
carrier frequency. The pulse shaping filter could be a rectangular impulse-response filter or a bandlimited square-root raised cosine filter.

Since each QAM symbol lasts over \( T_N \) seconds, all the inphase/quadrature carriers \( \{ \cos(2\pi f_n t + \phi_n), \sin(2\pi f_n t + \phi_n) \}_{n=0}^{N-1} \) can be made noncoherently orthogonal over the duration of \( T_N \) if the minimum carrier spacing is \( \Delta f = 1/T_N \), i.e., \( f_n = f_0 + n/T_N, \ n = 0, 1, \ldots, N - 1 \). Now, let the bandwidth of each subchannel is approximated as \( 1/T_N \). Then, the bandwidth of the OFDM signal in (1) is

\[
B = \frac{1}{T_N} + (N-1)\Delta f = \frac{1}{T_N} + (N-1)\frac{1}{T_N} = \frac{N}{T_N}.
\]

Figure 1 sketches the spectra of individual transmitted QAM signals. Note that the spectra of adjacent QAM signals overlap in frequency, but these signals are still orthogonal in time over the duration of \( T_N \) as long as the carrier spacing is chosen as \( \Delta f = 1/T_N \). It should also be pointed out that, being a summation of \( N \) QAM signals, the OFDM signal in (1) occupies a much wider frequency band, from approximately \( f_0 - \frac{1}{2T_N} \) to \( f_{N-1} + \frac{1}{2T_N} \). Thus, the OFDM signal is transmitted through a wideband channel whose frequency response \( H(f) \) is also depicted in Figure 1.

For a wideband channel, the frequency response \( H(f) \) typically exhibits a noticeable variation over the frequency, i.e., the channel is frequency-selective. The degree of frequency selectivity of a channel is usually measured by the so-called channel’s coherence bandwidth, denoted by \( B_c \). Loosely speaking, \( B_c \) is the frequency duration over which the channel frequency response is approximately constant. A related parameter that characterizes the frequency-selectivity of a channel is the multipath delay spread, which is loosely defined as the inverse of the coherence bandwidth, i.e., \( T_m = 1/B_c \). The meaning of this parameter is that, if a short transmitted pulse of duration \( T \) is transmitted over the channel, then the received signal will have a duration of approximately \( T + T_m \). If \( T_m \) is comparable to \( T \), then the consequence of delay spread is to cause inter-symbol interference (ISI) of the transmitted signal at the channel output.

For each QAM subchannel in OFDM, its bandwidth is \( B_N = \frac{1}{T_N} \). Thus, if \( B_N \) is made to be smaller than \( B_c \) (by using as many number of carriers as required), then each QAM subchannel experiences a frequency-flat channel (also called flat-fading channel). The condition \( B_N < B_c \) implies that \( T_N > T_m \), thus ISI can be made negligible. Flat fading and zero-ISI are desirable and this is achieved with OFDM transmission by using a large number of subcarriers.

The structure of the receiver that corresponds to the OFDM demodulator in Figure 2 is illustrated in Figure 3. Note that in the time domain, the wideband channel is represented by an impulse response \( h(t) \). However, by the preceding analysis, the complex value (a pair of real values, one for the inphase and one for the quadrature component) at the output of the matched filters is related to the complex QAM transmitted symbol as:

\[
\hat{X}_n[k] = H(f_n) \cdot X_n[k] + W_n[k]
\]

In (3), \( W_n[k] = W_n^{(I)}[k] + jW_n^{(Q)}[k] \) is a complex Gaussian random variable. Its real and imaginary parts are two independent and identically (i.i.d.) Gaussian random variables,
QAM demodulator on 0th subchannel

\[
\hat{X}_n[k] = \sum_{t} s(t) + h(t) + w(t) e^{-j2\pi(f_n-t)T_N} p(t)\star t = kT_N
\]

QAM symbol demapper

Parallel-to-serial converter

Detected infor. bits

\[
\vdots
\]

QAM demodulator on \((N-1)\)th subchannel

Figure 3: OFDM receiver built as multiple QAM demodulators.

Each with zero mean and variance \(N_0B_N/2\). If the channel gain \(H(f_n)\) in (3) is known\(^2\), then the detection of the transmitted QAM symbol from \(\hat{X}_n[k]\) is basically the same as that for an ideal AWGN channel.

It is important to point out that the simple and favourable relationship in (3) is achieved because of the following two main reasons: (i) the subchannels are orthogonal (though they overlap in frequency), and (ii) the number of subchannels is selected large enough so that each subchannel undergoes flat fading, hence there is negligible or zero ISI. There are, however, implementation issues corresponding to these two reasons. First, the orthogonality of subchannels is compromised by timing and frequency offsets. These offsets, even when relatively small, can significantly degrade performance, as they cause subchannels to interfere with each other. Second, when the number of subchannels is large, the requirement for multiple oscillators (both at the transmitter and receiver) makes the system very costly.

The next section presents a different implementation of OFDM systems that is based on the discrete Fourier transform (DFT) and inverse DFT (IDFT). As will be seen, such implementation requires only one oscillator at the transmitter and receiver. Furthermore, though not discussed in this note, the implementation with DFT/IDFT allows one to use many effective techniques for channel estimation and correction of timing and frequency offsets.

\(^2\)In practice, this is done by using a channel estimation/sounding algorithm.
3 Implementation of OFDM with DFT/IDFT

3.1 Review of the DFT

Let $x[n]$, $0 \leq n \leq N - 1$, be a discrete-time sequence. The $N$-point DFT of $x[n]$ is

$$\text{DFT}\{x[n]\} = X[i] \triangleq \sum_{n=0}^{N-1} x[n]e^{-j \frac{2\pi ni}{N}}, \quad 0 \leq i \leq N - 1. \quad (4)$$

The DFT is the discrete-time equivalent to the continuous-time Fourier transform, as $X[i]$ characterizes the frequency content of the time samples $x[n]$. Given $X[i]$, $0 \leq i \leq N - 1$, the sequence $x[n]$ can be recovered using the IDFT:

$$\text{IDFT}\{X[i]\} = x[n] \triangleq \frac{1}{N} \sum_{i=0}^{N-1} X[i]e^{j \frac{2\pi in}{N}}, \quad 0 \leq n \leq N - 1. \quad (5)$$

When $N$ is a power of two, the DFT and IDFT can be efficiently performed in hardware using the fast Fourier transform (FFT) and inverse FFT (IFFT) algorithms.

When the discrete-time sequence $x[n]$ is passed through a discrete-time linear time-invariant system whose impulse response is $h[n]$, the output $y[n]$ is the discrete-time convolution of the input and the channel impulse response. That is,

$$y[n] = h[n] * x[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad (6)$$

On the other hand, the $N$-point circular convolution of $x[n]$ and $h[n]$, both with length $N$, is defined as

$$y[n] = h[n] \otimes x[n] = x[n] \otimes h[n] = \sum_{k=0}^{N-1} h[k]x[(n-k) \mod N]. \quad (7)$$

From the definition of the DFT, the below derivation shows that circular convolution in time
leads to multiplication in frequency:

\[
\text{DFT}\{y[n]\} = x[n] \otimes h[n]
\]

\[
= \sum_{n=0}^{N-1} \left( \sum_{k=0}^{N-1} h[k] x[(n - k) \mod N] \right) e^{-j \frac{2\pi n i}{N}} = \sum_{k=0}^{N-1} h[k] \left( \sum_{n=0}^{N-1} x[(n - k) \mod N] e^{-j \frac{2\pi n i}{N}} \right)
\]

\[
= \sum_{k=0}^{N-1} h[k] \left( \sum_{l=-k}^{N-1-k} x[l \mod N] e^{-j \frac{2\pi (l+k) i}{N}} \right) = \sum_{k=0}^{N-1} h[k] \left( \sum_{l=-k}^{N-1-k} x[l \mod N] e^{-j \frac{2\pi l i}{N}} \right) e^{-j \frac{2\pi k i}{N}}
\]

\[
= \sum_{k=0}^{N-1} h[k] \left( x[N-k] e^{-j \frac{2\pi (N-k) i}{N}} + \ldots + x[N-1] e^{-j \frac{2\pi (N-1) i}{N}} + \sum_{l=0}^{N-1-k} x[l] e^{-j \frac{2\pi l i}{N}} \right) e^{-j \frac{2\pi k i}{N}}
\]

\[
= \sum_{k=0}^{N-1} h[k] \left( \sum_{l=0}^{N-1} x[l] e^{-j \frac{2\pi l i}{N}} \right) e^{-j \frac{2\pi k i}{N}}
\]

\[
= \sum_{k=0}^{N-1} h[k] X[i] e^{-j \frac{2\pi k i}{N}} = X[i] \left( \sum_{k=0}^{N-1} h[k] e^{-j \frac{2\pi k i}{N}} \right) = X[i] H[i], \quad 0 \leq i \leq N-1 \quad (8)
\]

### 3.2 OFDM Implementation

The implementation of OFDM transmitter and receiver is shown in Figure 4. The information bit stream is processed by a QAM mapper, resulting in a complex symbol stream \(X[0], X[1], \ldots, X[N-1]\). This symbol stream is passed through a serial-to-parallel converter, whose output is a set of \(N\) parallel QAM symbols \(X[0], \ldots, X[N-1]\), each symbol is transmitted over each of the \(N\) subcarriers. Thus, the \(N\) symbols from the serial-to-parallel converter correspond to the discrete frequency components of the OFDM signal \(s(t)\). In order to generate \(s(t)\), these frequency components are converted into time samples by the inverse DFT (which could be efficiently implemented using the IFFT algorithm if \(N\) is a power of 2). The IFFT yields an **OFDM symbol** consisting of the complex-valued sequence \(x[0], \ldots, x[N-1]\) of length \(N\), where

\[
x[n] = \frac{1}{N} \sum_{i=0}^{N-1} X[i] e^{j \frac{2\pi n i}{N}}, \quad 0 \leq n \leq N-1. \quad (9)
\]

Next, a very important operation called **cyclic prefix** (CP) extension is performed on the OFDM symbol to produce time samples

\[
\{\tilde{x}[n]\} = \{\tilde{x}[-\mu], \ldots, \tilde{x}[-1], \tilde{x}[0], \ldots, \tilde{x}[N-1]\} = \{x[N-\mu], \ldots, x[N-1], x[0], \ldots, x[N-1]\}. \quad (10)
\]

Let \(T_N\) be the duration of one OFDM symbol, then the distance between two adjacent time samples is \(T_s = T_N/N\). With \(\mu\) being the number of time samples in the cyclic prefix, the
length of the cyclic prefix is \( \mu T_s = \frac{\mu}{N} T_N \). The purpose of this operation as well as what determines the length \( \mu \) of the cyclic prefix shall be discussed shortly. The operation of CP extension is graphically illustrated in Figure 5.

Figure 5: Illustration of cyclic prefix extension.

After CP extension, these time samples are ordered by the parallel-to-serial converter and passed through a shaping filter, resulting in the complex baseband OFDM signal \( \tilde{x}(t) = \tilde{x}_I(t) + j\tilde{x}_Q(t) \). The baseband OFDM signal is then upconverted to frequency \( f_0 \). The transmitted signal is filtered by the channel impulse response \( h(t) \) and corrupted by additive white Gaussian noise \( w(t) \). The received signal is \( r(t) = s(t) * h(t) + w(t) \). This signal is downconverted to baseband and filtered to remove the high frequency components. The output of the matched filter is sampled every \( T_s \) to obtain \( y[n] \), \( -\mu \leq n \leq N - 1 \).
It is of interest to obtain a relationship between the set of “received” sample values \( y[n] \) and the set of “transmit” time samples \( \tilde{x}[n] \), \(-\mu \leq n \leq N - 1\). This is done as follows. Recall that the channel impulse response \( h(t) \) is characterized by a coherence bandwidth \( B_c \) and delay spread \( T_m \). While it is simple to use a large number of carrier \( N \) to make \( T_N > T_m \) (or equivalently \( B_N < B_c \)) so that each subchannel in OFDM experiences flat fading, as far as the duration between adjacent time samples \( \tilde{x}[n] \) and the sampling period used to obtain \( y[n] \) is concerned, it is clear that \( T_s = T_N/N \) can be smaller than \( T_m \), causing a significant “inter-symbol” interference in each of the value of \( y[n] \). Let \( \mu = \left\lceil \frac{T_m}{T_N} \right\rceil \). Then the equivalent discrete-time channel that defines the relationship between \( y[n] \) and \( \tilde{x}[n] \) can be represented by the impulse response \( \{ h[0], h[1], \ldots, h[\mu] \} \). This is illustrated in Figure 6.

Figure 6: Equivalent discrete-time channel.

By the operation of the CP insertion, one has \( \tilde{x}[n] = x[n \mod N] \) for \(-\mu \leq n \leq N - 1\), which also means that \( \tilde{x}[n - k] = x[(n - k) \mod N] \) for \(-\mu \leq n - k \leq N - 1\). Thus, given \( \tilde{x}[n] \) as the input of the channel and ignoring the noise component, the channel output between \( 0 \leq n \leq N - 1 \) can be computed as:

\[
\begin{align*}
y[n] &= \tilde{x}[n] \ast h[n] = \sum_{k=0}^{\mu} h[k] \tilde{x}[n - k] \\
&= \sum_{k=0}^{\mu} h[k] x[(n - k) \mod N] = x[n] \otimes h[n] \tag{11}
\end{align*}
\]

where the third equality follows from the fact that, for \( 0 \leq k \leq \mu \), \( \tilde{x}[n-k] = x[(n-k) \mod N] \) for \( 0 \leq n \leq N - 1 \).

The above analysis shows that, by appending a cyclic prefix to the channel input, the linear convolution associated with the channel impulse response becomes a circular convolution. Taking into account AWGN at the channel input, the DFT of the channel output in the yields

\[
Y[i] = \text{DFT}\{y[n] = (x[n] + w[n]) \otimes h[n]\} = X[i]H[i] + W[i], \quad 0 \leq i \leq N - 1, \tag{12}
\]

where \( W[i] \) is Gaussian noise component. Note that \( y[n] \), has length \( N + \mu \), yet from (11) the first \( \mu \) samples \( y[-\mu], \ldots, y[-1] \), are not needed. This is due to the redundancy associated with the cyclic prefix. Moreover, if we assume that the input \( x[n] \) is divided into data blocks of size \( N \) with a cyclic prefix appended to each block to form \( \tilde{x}[n] \), then the first \( \mu \) samples of \( y[n] = h[n] \ast \tilde{x}[n] \) in a given block are corrupted by ISI associated with the last \( \mu \) samples of \( x[n] \) in the previous block. The cyclic prefix serves to eliminate ISI between the data blocks (i.e., OFDM symbols) since the first \( \mu \) samples of the channel output affected by this
ISI can be discarded without any loss relative to the original information sequence. This is graphically illustrated in Figure 7.

The benefits of adding a cyclic prefix come at a cost. Since $\mu$ symbols are added to the input data blocks, there is an overhead of $\mu/N$ and a resulting data-rate reduction of $N/(\mu+N)$. The transmit power associated with sending the cyclic prefix is also wasted since this prefix consists of redundant data. Based on the relationship in (11) and (12), the prefix of $y[n]$ consisting of the first $\mu$ samples is removed. The remaining time samples are serial-to-parallel converted and passed through an FFT. This results in scaled versions of the original QAM symbols, $H[i]X[i]$, where $H[i] = H(f_i)$ is the flat-fading channel gain associated with the $i$th subchannel. The FFT output is parallel-to-serial converted and passed through a QAM demodulator to recover the original data.

The implementation of OFDM discussed in this section effectively decomposes the wideband channel $H(f)$ into a set of narrowband orthogonal subchannels with a different QAM symbol sent over each subchannel. Knowledge of the channel gains $H[i]$, $i = 0, \ldots, N-1$, is not needed for this decomposition. However, the demodulator needs the values of these channel gains to recover the original QAM symbols by dividing out these gains: $X[i] = Y[i] / H[i]$. This process is called frequency equalization. It is pointed out that, frequency equalization leads to noise enhancement, since the noise in the $i$th subchannel is also scaled by $1/H[i]$.

### 4 Application Example of OFDM: The IEEE 802.11a Wireless LAN Standard

The IEEE 802.11a Wireless LAN standard, which occupies 20 MHz of bandwidth in the 5 GHz unlicensed band, is the first version of 802.11 family that is based on OFDM. The 802.11g standard is virtually identical, but operates in the smaller and more crowded 2.4 GHz unlicensed ISM band. IEEE 802.11ac, released in December 2012, also operates in the 5 GHz band. It uses a wider RF bandwidth (80 or 160 MHz), multiple-input multiple-output (MIMO) technology (up to 8 MIMO streams), high-density modulation (up to 256 QAM) and could deliver a maximum data rate close to 7Gbps (on eight 256-QAM channels,
each delivering 866.7Mbps). The latest development of 802.11 standard is 802.11ad, which operates in the tri-band 2.5/5.0/60 GHz. This section discusses the properties of OFDM design used in 802.11a standard and some of the design choices.

In 802.11a standard, $N = 64$ subcarriers are generated. However, only 48 carriers are actually used for data transmission, the outer 12 carriers are zeroed in order to reduce adjacent channel interference, and 4 carriers used as pilot symbols for channel estimation and synchronization. The cyclic prefix consists of $\mu = 16$ samples, so the total number of samples associated with each OFDM symbol, including both data samples and the cyclic prefix, is 80. The transmitter gets periodic feedback from the receiver about the packet error rate, and uses this information to pick an appropriate error correction code and modulation scheme. The same code and modulation must be used for all the subcarriers at any given time. The error correction code is a convolutional code with one of three possible code rates: $r_c = 1/2$, $2/3$, or $3/4$. The modulation types that can be used on the subchannels are BPSK, QPSK, 16-QAM, or 64-QAM.

The bandwidth $B$ (and sampling rate $1/T_s$) is 20 MHz, and there are 64 subcarriers evenly spaced over that bandwidth. Therefore the subcarrier bandwidth is:

$$B_N = \frac{20 \text{ MHz}}{64} = 312.5 \text{ KHz.}$$  \hspace{1cm} (13)

$$B_N = \frac{20 \text{ MHz}}{64} = 312.5 \text{ KHz.}$$  \hspace{1cm} (13)

Since $\mu = 16$ and $1/T_s = 20$MHz, the maximum delay spread for which ISI is removed is

$$T_m < \mu T_s = \frac{16}{20 \text{ MHz}} = 0.8 \mu \text{sec,}$$  \hspace{1cm} (14)

which corresponds to delay spread in an indoor environment. Including both the data and cyclic prefix, there are 80=64+16 samples per OFDM symbol. Thus the symbol time per subchannel is

$$T_N^{(+\text{CP})} = T_N + \mu T_s = (N + \mu)T_s = 80T_s = \frac{80}{20 \times 10^6} = 4 \mu \text{sec} \hspace{1cm} (15)$$

The data rate per subchannel is $\log_2 M/T_N^{(+\text{CP})}$. Thus, the minimum data rate for this system, corresponding to BPSK (1 bit/symbol), an $r = 1/2$ code, and taking into account that only 48 subcarriers actually carry information data, is given by

$$(r_b)_{\text{min}} = 48 \text{ sub.} \times \frac{1/2 \text{ bit}}{\text{coded bit}} \times \frac{1 \text{ code bit}}{\text{sub. symbol}} \times \frac{1 \text{ sub. symbol}}{4 \times 10^{-6}} = 6 \text{Mbps} \hspace{1cm} (16)$$

The maximum data rate corresponds to 64-QAM and $r = 3/4$ code. It is given by,

$$(r_b)_{\text{max}} = 48 \text{ sub.} \times \frac{3/4 \text{ bit}}{\text{coded bit}} \times \frac{6 \text{ code bit}}{\text{sub. symbol}} \times \frac{1 \text{ sub. symbol}}{4 \times 10^{-6}} = 54 \text{Mbps} \hspace{1cm} (17)$$

Naturally, a wide range of data rates between these two extremes is possible.
5 Problems

1. Find the data rate of an 802.11a system assuming 16-QAM modulation and rate-2/3 channel coding.

2. Now consider a channel with a delay spread of $T_m = 20 \ \mu\text{sec}$. You want to design an OFDM system for use with such a channel so that the subchannel bandwidth is $B_N = B_c/2$, where $B_c$ is the coherence bandwidth of the channel.

   (a) Suppose that a square-root raised-cosine pulse with $\beta = 0.35$ is used and there are 128 carriers in your design. Using the approximation of $B_c \approx 1/T_m$, determine the total bandwidth occupied by the OFDM system.

   (b) Assuming a constant SNR $E_s/N_0 = 20 \ \text{dB}$ on each subchannel, find the maximum constellation size for $M$-QAM that can be sent over each subchannel with a target symbol error probability of $10^{-3}$. Find the corresponding total data rate (bits/sec) and bandwidth efficiency (bits/sec/Hz) of the system. Assume that $M$ is restricted to be a power of 2 and use the upper-bound in Equation (8.48) for estimating the symbol error probability of $M$-QAM.

3. Consider using OFDM technique over an AWGN channel, where there is no multipath problem and hence no need to use cyclic prefix or zero padding. Let $N$ be the number of orthogonal subcarriers employed and assume that $M$-QAM modulation is used for each subcarrier.

   (a) Determine the bandwidth efficiency, i.e., $r_b/W$.

   (b) Simply argue (without any derivation) what is the symbol error probability of the system. Together with the result in (a) what do you say about the representation of $M$-QAM OFDM on the bandwidth-power plane?

4. DOCSIS 3.1 (Data Over Cable Service Interface Specifications) is the latest standard (issued in October 2013) in providing high-speed data services via cables. Compared to DOCSIS 3.0, the most important technological change in DOCSIS 3.1 is the adoption of OFDM (as opposed to single-carrier QAM used in DOCSIS 3.0).

   For upstream transmission, i.e., from a cable modem (CM) to a cable modem termination system (CMTS), DOCSIS 3.1 uses OFDMA (orthogonal frequency-division multiple access), which is simply a multi-user version of OFDM that assigns subsets of subcarriers to individual CMs. Some relevant upstream OFDMA parameters specified in DOCSIS 3.1 are summarized below

   - The sampling rate, i.e., the rate at which the time sample values (after IFFT and CP extension) are transmitted is $R_s = 1/T_s = 102.4 \ \text{MHz}$.
   - There are two transmission modes that use different IFFT/FFT sizes: The 2k-mode has $N = 2048$, while the 4k-mode has $N = 4096$.
   - The subcarrier spacing, $\Delta f$, is 50kHz for the 2k-mode, while it is 25kHz for the 4k-mode.
For both transmission modes, the CP length is selected dynamically from a set of 11 different values. The minimum CP length is 96 time samples and the maximum CP length is 640 time samples.

Now for the questions:

(a) Consider the 2k-mode. Although the size of IFFT/FFT is 2048, the number of active subcarriers specified in the standard is 1901 (i.e., many values at the input of the IFFT block are set to zero). Determine the total “active” bandwidth of the upstream channel.

(b) What is the maximum delay spread, $T_m$, in seconds, that can be tolerated with zero-ISI in the upstream channel? Find the corresponding minimum value of the coherence bandwidth, $B_c$, of the upstream channel.

(c) The channel coding used for upstream transmission is a low-density parity-check (LDPC) code\(^3\) whose code rate is either 8/9, 28/33, or 3/4. Furthermore, the modulation constellations could range from BPSK to 4096-QAM. Assuming that all 1901 subcarriers are used for data, determine the maximum upstream transmission rate (in bits/sec) that can be achieved with DOCSIS 3.1. What is the corresponding bandwidth efficiency measure (in bits/sec/Hz)?

(d) Now, assume that one CM (i.e., one customer) is allocated 100 subcarriers. It is also known that 60 of these subcarriers experience a constant SNR = $E_s/N_0 = 20$dB and the remaining 40 subcarriers experience a constant SNR = $E_s/N_0 = 24$dB. Furthermore, it is known that the rate-3/4 LDPC code is used and the CP length is 256 time samples. Determine the total data rate (bits/sec) of this user under a target symbol error probability of $10^{-3}$.

5. Write a Matlab program to simulate the OFDM design of IEEE 802.11a. The program should include the OFDM transmitter, equivalent discrete-time channel, AWGN and OFDM demodulator. For the equivalent discrete-time channel, generate the channel filter coefficients \{\(h[n]\)\}_{n=0}^{\infty} as i.i.d. zero-mean complex Gaussian random variables, with variance 1/2 for real and imaginary parts. For the OFDM demodulator you can assume that the channel coefficients \{\(h[n]\)\}_{n=0}^{\infty} are perfectly known. Test your Matlab program with 64-QAM modulation on all data subcarriers and obtain a plot of the bit error rate versus received $E_b/N_0$.

\(^3\)LDPC codes are basically long block codes that can be efficiently decoded with iterative decoding algorithms.