EE480.3 Digital Control Systems

Part 8. Root Locus Method
- using the z-transform

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Controller Design by Root Locus Method

The root locus method is a method to determine gain $K$ such that the dominant poles of unity gain closed loop system come close to a desired location(s).

When the adjustable gain $K$ is applied to an open loop transfer function $G(z)$, the closed loop transfer function is given by

$$G_c(z) = \frac{KG(z)}{1 + KG(z)}.$$  

The characteristic equation $1 + KG(z) = 0$ requires that

$$KG(z) = 1 \mp 180^\circ(2k + 1), \quad \text{where } k \text{ is an integer.}$$
Example - Drawing the root locus

The procedure to plot the root locus of a given system $G(z)$ is the same as you did for a continuous time system $G(s)$.

- Find the root locus on the real axis. Make pairs of poles or zeros (mixed) from $(1,0)$ to $-\infty$.
- Calculate the angles of asymptotes, $\pm 180/(p - z)$.
- Find break-in or break-away points. \[ \frac{dK}{dz} = \frac{d}{dz} \left( -\frac{1}{G(z)} \right) = 0. \]
From the angle condition, \(0.5 \leq z \leq 1\) is part of the root locus on the real axis. For a complex \(z\) satisfying the above condition, let \(z = \alpha + j\beta\).

\[
\angle G(z) = \angle z - \angle(z - 1) - \angle(z - 0.5)
\]

\[
= \arctan \frac{\beta}{\alpha} - \arctan \frac{\beta}{\alpha - 1} - \arctan \frac{\beta}{\alpha - 0.5}
\]

\[
\tan\{\arctan \frac{\beta}{\alpha} - \arctan \frac{\beta}{\alpha - 1} - \arctan \frac{\beta}{\alpha - 0.5}\} = \tan 180^\circ = 0
\]

Using the identity,

\[
\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},
\]

For this particular example, we obtain \(\alpha^2 + \beta^2 = 0.5\), the equation of a circle with radius \(\sqrt{0.5}\).
Writing the characteristic equation $1 + KG(z) = 0$ for $K$,

$$K = -\frac{1}{G(z)} = \frac{(z - 1)(z - 0.5)}{z}$$

$$\frac{dK}{dz} = -\frac{(z - 0.707)(z + 0.707)}{z^2} = 0$$

gives the break away and break in points as $z = \pm 0.707$.

The number of poles $p = 2$ and the number of poles $z = 1$ yields

$$\pm 180/(p - z) = \pm 180/(2 - 1) = \pm 180^\circ.$$
Root Locus in z-plane
% Root Locus Example of z-plane Root Locus
zeros=[0]
poles=[0.5, 1]
num=poly(zeros);
den=poly(poles);
zrltemp1;
rlocus(num, den);
[r,k]=rlocus(num, den)
axis([-1.0, 1.0, -1.0, 1.0]);
axis square;
Root Locus Template for z-plane

The familiar grids in parallel with the real and imaginary axis of s-plane represent vertically the frequency, horizontally the inverse of time constant. However, this does not apply to the z-plane. The z-plane is mapped from the s-plane by

\[ z = e^{sT} \quad \text{where} \quad s = \sigma + j\omega \]

\[ z = e^{sT} = e^{\sigma T} e^{j\omega T} = e^{\sigma T} \angle \omega T = e^{\omega T} \angle 2\pi \frac{f}{f_s} \]
A strip zone in the LHP s-plane between $\pm \omega_s/2$ corresponds to the inside of the unit circle in the z-plane. Note that z-plane is periodic with respect to $\omega T$. 

\[ z = e^{sT} = e^{\sigma T} e^{j\omega T} \]
The standard guide lines used in s-plane such as constant damping ratio, constant $\sigma$, constant $\omega_d$ are mapped by $z = e^{sT}$.

- **ζ damping ratio constant**

  From the standard form of the second order system,

  $$ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} $$

  Poles are:  
  $$ s = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} $$

  Damping ratio is given by $\zeta = \cos \theta$. For a constant damping ratio, $\theta = \text{const}$, and its slope is $-\sqrt{1 - \zeta^2} / \zeta$. Letting $s = \sigma + j\omega$, the constant $\zeta$ in s-plane is given by

  $$ \omega = -\frac{\sqrt{1 - \zeta^2}}{\zeta} \sigma $$

  Therefore, in z-plane,

  $$ z = e^{sT} = e^{\sigma T} e^{-j \frac{\sqrt{1 - \zeta^2}}{\zeta} \sigma T} $$
When $\sigma \to -\infty$, $z$ rotates counter clockwise and the radius shrinks. $\zeta = 0, 0.1, 0.2 \cdots 0.9$ are drawn in the template.

- $\sigma = -\zeta \omega_n = -1/T_c$ (time constant) constant

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}$$

If $\omega$ varies while $\sigma$ stays as constant, the locus is a circle. $e^{\sigma} = 1.0, 0.8, \cdots 0.2, 0$ are drawn in the template.

- $\omega_d$ damped natural frequency constant

Since $s = \sigma + j\omega_d$ and $\omega_d = \omega_n \sqrt{1 - \zeta^2}$,

$$z = e^{sT} = e^{\sigma T} e^{j\omega_d T} = e^{\sigma T} \angle \omega_d T$$

Therefore, $\omega_d$ constant is angle $\omega_d T$ constant. There are 10 lines $\omega T \frac{k}{10}$ drawn in the template.

- $\omega_n$ natural frequency constant
\[ \omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} \]

When damping ratio \( \zeta = 0 \), \( \omega_n = \omega_d \). Since \( \omega_n \) is a function of \( \zeta \), the locus of \( \omega_n = \text{constant} \) is a curve that asymptotically approaching a \( \omega_d \) constant line.

- **Percent Overshoot**

\[ M_p = 100 \times e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \]

- **Settling Time**

\[ t_s = \frac{3}{\zeta \omega_n} = 3T_c \quad \text{for 5\% criterion} \]

\[ t_s = \frac{4}{\zeta \omega_n} = 4T_c \quad \text{for 2\% criterion} \]
\[ z = \text{plane loci of roots of constant } \zeta \text{ and } \omega_n \]
\[ s = -\frac{1}{\omega_n} \pm j/\omega_n \sqrt{1 - \zeta^2} \]
\[ z = e^{j\theta} \]
\[ T = \text{sampling period} \]
\( z = \text{plane loci of roots of constant } \zeta \text{ and } \omega_n \)

\[ z = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \]

\[ z = e^{j\theta} \]

\( T = \text{sampling period} \)
$z = \text{plane loci of roots of constant } \xi \text{ and } \omega_n$

$s = -\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2}$

$z = e^{j\xi}$

$T = \text{sampling period}$
$z = \text{plane loci of roots of constant } \xi \text{ and } \omega_n$

$s = -\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2}$

$z = e^{j\omega_n \xi}$

$T = \text{sampling period}$
Consider this cruise control system for automobile. Ignore the sensor gain of 0.03 and A/D in this block diagram. Feedback is unity gain feedback, and the analog error is sampled before it goes to computer.
Examine if the unity gain feedback system can achieve the time constant of 2 second, and damping ratio $\zeta > 0.4$ by adjusting the forward gain $K$ (Assume the computer sets the gain $K$).

The analog system of the cruise control system is

$$G(s) = \frac{6.4}{(s + 1)(s + 0.2)}.$$  

The discrete time transfer function (open loop) is

$$G_d(z) = \frac{0.1183z + 0.1092}{z^2 - 1.78z + 0.7866} = 0.1183 \frac{z - 0.9231}{(z - 0.9608)(z - 0.8187)}.$$  

From $\sigma = -\frac{1}{T_c} = -2$,
the radius of constant $\sigma$ circle in the $z$-plane is

$$e^{\sigma T} = e^{-0.2} = 0.9048.$$
T=0.2
Sys=tf([6.4],poly([-1,-0.2]))
dSys=c2d(Sys,T,'zoh')
[dSyszeros,dSyspoles,Ksys,ts]=zpkdata(dSys,'v')
zeros=[dSyszeros']
poles=[dSyspoles']
num=poly(zeros)*Ksys;
den=poly(poles);
figure(1);
zrltempl;
rlocus(num, den) %Closed loop with gain K
[r,k]=rlocus(num, den);
axis([-1.0, 1.0, -1.0, 1.0]);
axis square;
cp=0.8061 + 0.5398*i
amp=abs(cp)
theta=atan(imag(cp)/real(cp))
zeta=-log(amp)/sqrt(log(amp)^2+theta^2)
r =
0.9608  0.8187
0.8955  0.8813
0.8884  0.8884
0.8884 - 0.0071i  0.8884 + 0.0071i
0.8860 - 0.0935i  0.8860 + 0.0935i
0.8815 - 0.1573i  0.8815 + 0.1573i
0.8719 - 0.2436i  0.8719 + 0.2436i
0.8511 - 0.3655i  0.8511 + 0.3655i
(0.8061 - 0.5398i  0.8061 + 0.5398i)
0.7085 - 0.7869i  0.7085 + 0.7869i
0.4973 - 1.1243i  0.4973 + 1.1243i

k =
Columns 1 through 7
0  0.0233  0.0235  0.0238  0.0643  0.1393  0.3017
Columns 8 through 14
0.6534 (1.4151)  3.0648  6.6374  10.5059  14.3745  18.5636

At the gain 1.4151, the closed loop poles are at $z = 0.8061 \pm 0.5398i$ which give the closest to $e^{\sigma T} = e^{-0.1} = 0.9$. The calculated radius is 0.9701. The damping ratio at this pole location is 0.0513 (less than 0.1) and gives significant overshoot. Therefore, simple gain adjustment cannot satisfy the required performance.
A digital PD controller represented by

\[
D(z) = K_p + K_d(1 - z^{-1}) = (K_p + K_d) \frac{z - \frac{K_d}{K_p + K_d}}{z}
\]

might be able to satisfy the conditions. \(K_p = 1\) and \(K_d = 2\) was tried in this MATLAB program. The z-plance root locus is shown below.
\[ r = \]
\[
\begin{array}{llllll}
0 & 0.9608 & 0.8187 \\
0.0088 & 0.8914 & 0.8684 \\
0.0089 & 0.8868 & 0.8726 \\
0.0090 & 0.8796 - 0.0000i & 0.8796 + 0.0000i \\
0.0091 & 0.8795 - 0.0071i & 0.8795 + 0.0071i \\
0.0198 & 0.8677 - 0.0754i & 0.8677 + 0.0754i \\
0.0459 & 0.8399 - 0.1343i & 0.8399 + 0.1343i \\
0.1146 & (0.7729 - 0.2067i) & 0.7729 + 0.2067i \\
0.2204 & 0.6839 - 0.2579i & 0.6839 + 0.2579i \\
0.3923 & 0.5618 - 0.3126i & 0.5618 + 0.3126i \\
0.5415 & 0.4474 - 0.4359i & 0.4474 + 0.4359i \\
0.5872 & 0.3846 - 0.5436i & 0.3846 + 0.5436i \\
0.6199 & 0.2885 - 0.7037i & 0.2885 + 0.7037i \\
0.6420 & 0.1010 - 0.9416i & 0.1010 + 0.9416i \\
\end{array}
\]
\[ k = \]
\[
\begin{array}{llllllllllllllll}
Columns 1 through 7 & \\
0 & 0.0932 & 0.0948 & 0.0958 & 0.0967 & 0.2061 & 0.4557 \\
Columns 8 through 14 & \\
(1.0077) & 1.6179 & 2.2281 & 2.9027 & 3.5774 & 4.9266 & 7.9100 \\
\end{array}
\]

The closed loop poles at 0.7729 ± 0.2067i at the gain of 1.0077 gives \( \zeta = 0.6492 \), and the radius is 0.8001.
The additional gain to be added to the open loop gain is 1.0077, which includes the gain from the PD controller $K_p + K_d = 2$, since only the PD controller’s pole and zero are considered to modify the system open loop transfer function. The following step response indicates approximately 10% overshoot, which is expected from $\zeta = 0.6492$. 
3 Steady State Error

\[ E = R - GE^* \]
\[ E^* = R^* - G^*E^* \]
\[ C = GE^* \]
\[ C^* = G^*E^* \]

Therefore,

\[ E^* = \frac{1}{1 + G^*} R^* \]
or
\[ E(z) = \frac{1}{1 + G(z)} R(z) \]
Offset due to Step Input

When unit step input, \( R(z) = \frac{1}{1 - z^{-1}} \) is applied,

\[
e_{ss} = \lim_{z \to 1} (z - 1)E(z) = \lim_{z \to 1} (z - 1) \frac{1}{1 + G(z)} \frac{1}{1 - z^{-1}}
\]

\[
= \lim_{z \to 1} \frac{z}{1 + G(z)} = \frac{1}{1 + \lim_{z \to 1} G(z)}
\]

Let position error coefficient \( K_p \) be

\[
K_p = \lim_{z \to 1} G(z).
\]

Then, \( e_{ss} = \frac{1}{1 + K_p} \).
In the previous example of a cruise control system, the open loop transfer function before introducing PD controller was

\[
G_d(z) = \frac{0.1183z + 0.1092}{z^2 - 1.78z + 0.7866} = 0.1183 \frac{z - 0.9231}{(z - 0.9608)(z - 0.8187)}
\]

The PD controller added one pole and one zero, and the root locus method gave an additional gain of \(Gain = 1.0077\). Therefore, the open loop transfer function with the PD controller,

\[
G_{PD}(z) = 1.0077 \frac{z - \frac{K_d}{K_p+K_d}}{z} = 1.0077 \frac{z - 2/3}{z}.
\]

The compensated open loop transfer function is,

\[
G(z) = 1.0077 \frac{z - 2/3}{z} \times 0.1183 \frac{z - 0.9231}{(z - 0.9608)(z - 0.8187)}
\]
Thus,

\[ e_{ss} = \frac{1}{1 + K_p} = 0.0851, \quad \text{where} \quad K_p = \lim_{z \to 1} G(z) \]

From simulation, the step response at time \( t = 50 \times 0.2 \) is \( 1-\text{angle}(50) = 0.0851 \). This PD controller introduces a steady state error (offset) of 8.51\%. 
Velocity Error due to Ramp Input

When unit ramp input, \( R(z) = \frac{Tz^{-1}}{(1 - z^{-1})^2} = \frac{Tz}{(z - 1)^2} \) is applied,

\[
e_{ss} = \lim_{z \to 1} (z - 1) \cdot \frac{1}{1 + G(z)} \cdot \frac{Tz}{(z - 1)^2}
\]

\[
= \lim_{z \to 1} \frac{Tz}{(z - 1) + (z - 1)G(z)}
\]

\[
= \lim_{z \to 1} \frac{T}{(z - 1)G(z)}
\]

Let velocity error coefficient \( K_v \) be

\[
K_v = \lim_{z \to 1} \frac{1}{T}(z - 1)G(z),
\]

Then, \( e_{ss} = \frac{1}{K_v} \).
If $G(z)$ is expressed in the form of

$$G(z) = \frac{1}{(z - 1)^N} G'(z),$$

$G(z)$ is called Type N system. For $N = 0$, the system is of Type 0. Since

$$K_p = \lim_{z \to 1} \frac{1}{(z - 1)^N} G'(z) = \infty \quad \text{for } N \geq 1,$$

$$e_{ss} = \frac{1}{1 + K_p} \quad \text{for Type 0 system}$$

$$e_{ss} = 0 \quad \text{for Type } N \geq 1 \text{ system}$$

$$K_v = \lim_{z \to 1} \frac{1}{T} (z - 1) G(z) = \lim_{z \to 1} \frac{1}{T} (z - 1) \frac{1}{(z - 1)^N} G'(z)$$

$$e_{ss} = \frac{1}{K_v} = \infty \quad \text{for Type 0 system}$$
\[ e_{ss} = \frac{1}{K_v} = \text{const.} \quad \text{for Type 1 system} \]

\[ e_{ss} = \frac{1}{K_v} = 0 \quad \text{for Type } N \geq 2 \text{ system} \]

Note that the cruise control system is of Type 0. The steady state error \( e_{ss} \) for step input is constant that appears as an offset. The steady state error \( e_{ss} \) for ramp input is infinity that means the output will never catch up the input. Also, the observation tells that PD controller is not an effective means to reduce steady state errors. A pole at \( z = 1 \) should be introduced if steady state error is a concern.
4 Satellite Position Control System

A communication satellite must be exactly positioned to a specified angular position to maintain communication link to the earth station.

- A pair of bi-directional rockets are equipped on the outside wall of the satellite to make necessary corrections if disturbances offset the angular position.
- The satellite, as a control system, is a rotating object placed in a frictionless space.
- The torque exerted by a pair of the rockets is controlled by an electrical input voltage.
- The position information is available also in a form of electrical signal, which indicates the angle deviation relative to the perfect alignment for the best communication condition.
• Digital controller $D(z)$ of the first order phase lead type is used to stabilize the satellite position.

• A sampling rate of the digital controller is 10 samples per second, or $T_s = 0.1$ sec.

Fig. 1  Communication Satellite Position Stabilizer
Design Criteria

Our design considers the following three major criteria:

1. The damping ratio must be greater than 0.6, i.e. $\zeta = 0.6$. It is desirable to keep the dominant poles to be slightly underdamped.

2. The damped natural frequency should not exceed 1 Hz. The sampling rate of 10 samples/sec. has already been chosen taking this condition into account. One cycle of oscillatory responses yields at least 10 samples per cycle. This is equivalent to $\omega_d < 0.1\omega_s$.

3. The time constant of the closed loop control system should be less than 0.5 second. This condition of $T = 1/\zeta\omega_n = 0.5$ (sec) gives a settling time of $t_s = 3/\zeta\omega_n = 1.5$ (sec) for the 5% criterion.
The $z$-plane region satisfying the design criteria

1. $\zeta > 0.6$

2. Since $\omega_d < 0.1\omega_s = 0.1 \times \frac{2\pi}{T_s}$, $\omega_dT_s < 0.2\pi = 36^\circ$.

3. $T < 0.5 \Rightarrow \sigma = \frac{1}{T} > 2 \Rightarrow z = e^{-\sigma T} < e^{-2T} = e^{-0.2} = 0.8187$
$z = \text{plane loci of roots of constant } \zeta \text{ and } \omega_n$

$s = -\zeta \omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$

$z = e^{j\theta}$

$T = \text{sampling period}$
Transfer Functions

Let the forward and the feedback transfer function be defined as

\[
\tilde{G}(s) = \frac{1 - e^{-sT_s}}{s} \times \frac{0.02}{s^2}
\]  

(1)

\[
H(s) = 360.
\]  

(2)

Fig. 2  Symbolized Block Diagram
The input/output relationship of the discrete time closed loop system is

\[ C^* = \frac{D^* \bar{G}^*}{1 + D^* \bar{G} \bar{H}^*} R^*. \]
Using the residue method to find $G(z)$,

$$G(z) = 0.02 \ (1 - z^{-1}) \ Z[\frac{1}{s^3}]^* \quad (3)$$

$$= 0.02 \ (1 - z^{-1}) \ [ \text{Residue of} \ \frac{1}{\lambda^3(1 - e^{\lambda T_s z^{-1}})} \ \text{at} \ \lambda = 0] \quad (4)$$

$$= 0.02 \ \frac{T_s^2}{2} \ \frac{z + 1}{(z - 1)^2} \quad (5)$$
First Order Phase Lead/Lag compensator

Consider an analog first order compensator,

\[ D(s) = \frac{s + a}{s + b} \]

which is a phase lead compensator if \( b > a > 0 \) and a phase lag compensator if \( a > b > 0 \). The impulse invariant discrete transfer function \( D(z) \) is

\[
D(z) = Z[1 + \frac{a - b}{s + b}]^* \\
= Z[\delta(t) + (a - b)e^{-bt}]^* \\
= (1 + a - b) \frac{z - e^{-bT_s}}{z - e^{-bT_s}}/(1 + a - b)
\]

\[ (6), (7), (8) \]
Since the transformed $D(z)$ contains one zero and one pole, we will use the following general form for the discrete first order compensator $D(z)$,

$$D(z) = K \frac{z - z_0}{z - z_p}.$$ 

**Negative Feedback with a Compensator of Constant Gain $D(z) = K$**

By removing the pole $z = z_p$ and the zero $z = z_0$, the characteristic equation of this closed loop system is written as

$$1 + D^*(z)\bar{G}H^*(z) = 1 + 360 \times 0.02 K \frac{T_s^2}{2} \frac{z + 1}{(z - 1)^2} = 0.$$
Letting $K' = 360 \times 0.02 K \frac{T_s^2}{2} = 0.036 K$,

$$K' = \frac{(z - 1)^2}{z + 1}.$$  

From

$$\frac{d}{dz} K' = \frac{d}{dz} \frac{(z - 1)^2}{z + 1} = 0,$$

we can find two break points (one for break-away and the other for break-in) at $z = -3$ and $z = 1$. The root locus is a circle passing through these points. It is apparent that the closed loop system is unstable. This result agrees with a fact that applying negative feedback around two integrators always produces (undamped) oscillation.
Phase Lead Compensators

Incorporating the first order phase lead compensator

\[ D(z) = K \frac{z - z_0}{z - z_p} \]

into the system, the characteristic equation of the compensated closed loop system is

\[ 1 + K' \frac{T_s^2}{2} \frac{(z - z_0)(z + 1)}{(z - z_p)(z - 1)^2} = 0. \]

Keep in mind that lead compensators need to satisfy \( z_0 > z_p \), i.e. the zero \( z = z_0 \) on the left-hand side of the pole \( z = z_p \). A root locus can be drawn for this characteristic equation by varying a parameter \( K' \).
We consider the following three cases of the phase lead compensator to examine if all the required design criteria are met.

1. Case of $z_0 = 0.8$ and $z_p = 0.2$

2. Case of $z_0 = 0.9$ and $z_p = 0.2$

3. Case of $z_0 = 0.8$ and $z_p = -0.2$
Root Locus Plot

To draw root loci, we can use a Matlab routine `rlocus.m` which can be called by `rlocus(num, den)`. This subroutine calculates and plots the locus of the roots of:

\[ 1 + K \frac{NUM(z)}{DEN(z)} = 0 \]

for a set of gains \( K \) which are adaptively calculated to produce a smooth plot. `rlocus.m` is applicable for both s-plane and z-plane. A modified version `rloci.m` drawing root loci with a thicker line width and a program `zrltemp1.m` which produces a z-plane template are provided for your convenience.
Case I of $z_0 = 0.8$ and $z_p = 0.2$

Fig. 3 Root Locus Plot for Case I.
Case II of $z_0 = 0.9$ and $z_p = 0.2$

Fig. 4 Root Locus Plot for Case II.
**Case III** of $z_0 = 0.8$ and $z_p = -0.2$

Fig. 5 Root Locus Plot for Case III.
Verification of the Designed Control System

After examining the plotted root loci for all three cases, one possible implementation of the phase lead compensator is to use the third case with a gain of $K' = 0.4$. The compensator transfer function is, therefore,

$$ D(z) = K \frac{z - z_0}{z - z_p} = \frac{K'}{0.036} \frac{z - z_0}{z - z_p} $$

$$ = 11.1111 \frac{z - 0.8}{z + 0.2}. $$

The discrete transfer function was obtained earlier,

$$ C^* = \frac{D^* \tilde{G}^*}{1 + D^* \tilde{G}H^*} R^*. $$
If we use separate blocks for the satellite and the phase lead compensator, and apply the feedback as the original block diagram, simulation can be performed by SIMULINK of the MATLAB. The figures, Fig. 6 and Fig. 7, show the simulation diagram of the satellite position control and its result (response) to a step input that positions the satellite at 90° which is one quarter revolution.

Fig. 4 Simulation Diagram on SIMULINK.
Fig. 5 Step Response of the Closed Loop Satellite Position Control.
Finding $G(z)$ from $\tilde{G}(s)$ using MATLAB

%% C2DZP.M continuous (w/ZOH) pole-zero to discrete pole-zero
% [a,b,c,d]=zp2ss(z,p,ko);
% [phi,gam]=c2d(a,b,dt);
% [zd,pd,kd]=ss2zp(phi,gam,c,d,1);
%
% Satellite System is
% \[
% \frac{1}{s^2}
% \]
% G(s)=0.02 -----
% \[
% s^{-2}
% \]

dt=0.1; % sampling time
z=[]; % no zeros
p=[0, 0]; % two poles at 0
ko=0.02; % gain is 0.02
% -------- from c2dp ---------------------------------------
[a,b,c,d]=zp2ss(z,p,ko);
[phi,gam]=c2d(a,b,dt);
[zd,pd,kd]=ss2zp(phi,gam,c,d,1);
% ----------------------------------------------------------
disp('zeros in G(z) are:');
zd
disp('poles in G(z) are:');
pd
disp('gain of G(z) is:');
k =
zeros in G(z) are:
z =
-1

poles in G(z) are:
p =
1
1

gain of G(z) is:
k =
1.0000e-004

**Plotting Root Loci with MATLAB**

\%
\% Design Example of z-plane Root Locus Method
\% Case Study ... Satellite Position Control
\% -------------------------------------------
\% Case 1
figure(1);
z = [0.8 -1];
p = [1 1 0.2];
num=poly(zeros);
den=poly(poles);
zrltempl;
rloci(num, den);

% Case 2
figure(2);
zeros=[0.9 -1];
poles=[1 1 0.2];
num=poly(zeros);
den=poly(poles);
zrltempl;
rloci(num, den);

% Case 3
figure(3);
zeros=[0.8 -1];
poles=[1 1 -0.2];
num=poly(zeros);
den=poly(poles);
zrltempl;
rloci(num, den);

**Drawing z-Domain Root Locus Template**

% --------------------------------------------------------
% Plot z-plane root locus template
% K. Takaya, Elec. Eng. Univ. of Saskatchewan
% --------------------------------------------------------
hold off;
axis([-1.5, 1.5, -1.5, 1.5]);
axis('square');
theta=[0: pi/90: 2*pi];
cir=exp(j*theta);
for radius=1.0: -0.2: 0.2
    plot(radius*cir,'y'); hold on;
end
line=[0.0: 0.02: 1.0];
for angle=pi/10: pi/10: 2*pi
    plot(line*exp(j*angle),'y'); hold on;
end
sy=[0.0: pi/50: pi];
for zeta=0.1: 0.1: 0.9
    sx=(-zeta/sqrt(1-zeta^2))*sy;
    zzeta=exp(sx+j*sy);
    plot(zzeta,'y'); hold on;
    plot(conj(zzeta),'y'); hold on;
end
theta=[pi/2: pi/90: pi+pi/2];
for radius=0.1*pi: 0.1*pi: pi
natur=exp(radius*(cos(theta)+j*sin(theta)));  
plot(natur,'y'); hold on;  
end  
title('z-plane ROOT LOCI'); grid;  

Numerical Listing of Gain vs Poles for Case III.  

[K3,R3] =1.0e+002 *  
0.0004 0.0097 - 0.0011i 0.0097 + 0.0011i -0.0018  
0.0006 0.0095 - 0.0014i 0.0095 + 0.0014i -0.0016  
0.0010 0.0092 - 0.0017i 0.0092 + 0.0017i -0.0014  
0.0014 0.0089 - 0.0019i 0.0089 + 0.0019i -0.0011  
0.0018 0.0084 - 0.0021i 0.0084 + 0.0021i -0.0007  
0.0023 0.0080 - 0.0022i 0.0080 + 0.0022i -0.0002  
0.0028 0.0074 - 0.0022i 0.0074 + 0.0022i 0.0004  
0.0033 0.0067 - 0.0021i 0.0067 + 0.0021i 0.0013  
0.0037 0.0058 - 0.0017i 0.0058 + 0.0017i 0.0026  
0.0038 0.0057 - 0.0016i 0.0057 + 0.0016i 0.0029  
0.0038 0.0055 - 0.0014i 0.0055 + 0.0014i 0.0033  
0.0039 0.0051 - 0.0012i 0.0051 + 0.0012i 0.0039  
0.0039 0.0044 - 0.0013i 0.0044 + 0.0013i 0.0052  
0.0039 0.0042 - 0.0017i 0.0042 + 0.0017i 0.0057  
0.0040 0.0040 - 0.0020i 0.0040 + 0.0020i 0.0060  
0.0040 0.0039 - 0.0022i 0.0039 + 0.0022i 0.0062  
0.0041 0.0038 - 0.0024i 0.0038 + 0.0024i 0.0063  
0.0042 0.0036 - 0.0028i 0.0036 + 0.0028i 0.0065
<table>
<thead>
<tr>
<th>Value</th>
<th>Real Part</th>
<th>Imaginary Part</th>
<th>Real Part + Imaginary Part</th>
<th>Real Part - Imaginary Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0045</td>
<td>0.0033</td>
<td>-0.0036i</td>
<td>0.0033 + 0.0036i</td>
<td>0.0069</td>
</tr>
<tr>
<td>0.0055</td>
<td>0.0026</td>
<td>-0.0051i</td>
<td>0.0026 + 0.0051i</td>
<td>0.0073</td>
</tr>
<tr>
<td>0.0077</td>
<td>0.0014</td>
<td>-0.0073i</td>
<td>0.0014 + 0.0073i</td>
<td>0.0076</td>
</tr>
</tbody>
</table>
Assignment No. 7

This classnote Part-8 describes how to apply the root locus method to a discrete time model of the satellite attitude control system. Three cases of the phase lead compensator were demonstrated to examine if three design criteria can be met by adjusting the loop gain \( K \).

Try another first order phase lead compensator having a pole at \( z = 0 \) and a zero at \( z = 0.8 \) to see if this compensator can satisfy the same design criteria as used in the note. If it works, find the gain \( K \) which places the closed poles in the area that satisfies design criteria. Calculate the following values:

1. Closed loop poles for the gain \( K \) you chose.
2. Damping ratio, percent overshoot and time constant if the
system is underdamped.

3. Steady state error for the unit step input.

Plot the step response of the closed loop system by using SIMULINK of MATLAB.