EE481 Control Systems

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Course Outline & Laplace Transform, Modeling

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University of Saskatchewan, Electrical Engineering

EE 481.3 Control Systems

April 2009, Kunio Takaya

Textbook: Norman S. Nise, "Control Systems Engineering" Fifth Edition, John Wiley & Sons, Inc. 2008, ISBN-13 978-0471-79475-2.

Marks: Midterm Exam: 30%, Final Exam 55%, and Assignments 15%

1. Modeling in the frequency domain

- Laplace transform
- Transfer functions

2. Modeling in the time domain

- Linear differential equations
- State-Space representation

3. Time response
    - Second-Order Systems
    - Poles and zeros
    - Time domain solution of state equations

4. Reduction of multiple subsystems
    - Block diagrams and Signal-Flow graphs
    - Mason's rule
    - Similarity transformations

5. Stability
    - Routh-Hurwitz criterion
    - Stability in State-Space

6. Steady-State errors

- Steady-State error for unity gain feedback
- Steady-State error for disturbance

7. Root Locus techniques

- Sketching the root locus
- Transient response design via gain adjustment

8. Design via root locus

- Cascade compensation
- Improving transient response and steady-state error

9. Frequency response techniques

- Bode plots
- Nyquist diagrams
- Systems with time delay

10. Design via frequency response

- Lag compasation
- Lead compensation

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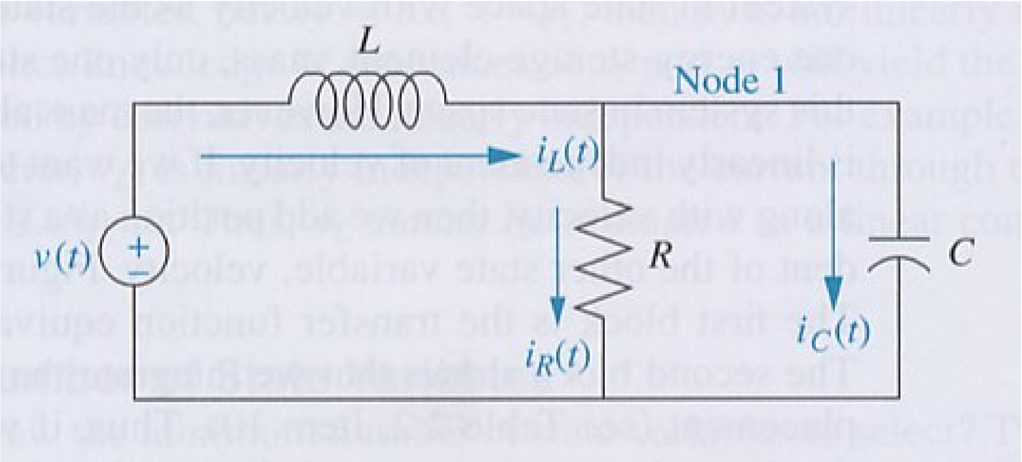
1 Modeling in the Time Domain

Electrical Example

Given the circuit below, find the current i_R through resistor R for a given input voltage v. Using the impedance,



$$
\begin{aligned}
I_R(s) &= \frac{V_R(s)}{R} = \frac{1}{R}\frac{\dfrac{R\cdot\dfrac{1}{sC}}{R+\dfrac{1}{sC}}}{sL+\dfrac{R\cdot\dfrac{1}{sC}}{R+\dfrac{1}{sC}}}V(s) \\
&= \frac{1}{R}\frac{R}{sL(sCR+1)+R}V(s) \\
&= \frac{1}{s^2CRL+sL+R}V(s)
\end{aligned}
$$

Using differential equations,

$$
\begin{aligned}
C\frac{dv_C}{dt} &= -\frac{1}{R}v_C + i_L \\
L\frac{di_L}{dt} &= -v_C + v(t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dv_C}{dt} &= -\frac{1}{RC}v_C + \frac{1}{C}i_L \\
\frac{di_L}{dt} &= -\frac{1}{L}v_C + \frac{1}{L}v(t)
\end{aligned}
$$

$$
\text{the output is} \quad i_R = \frac{1}{R}v_C
$$

This set of simultanious differential equation is written in a vector form as a state equation,

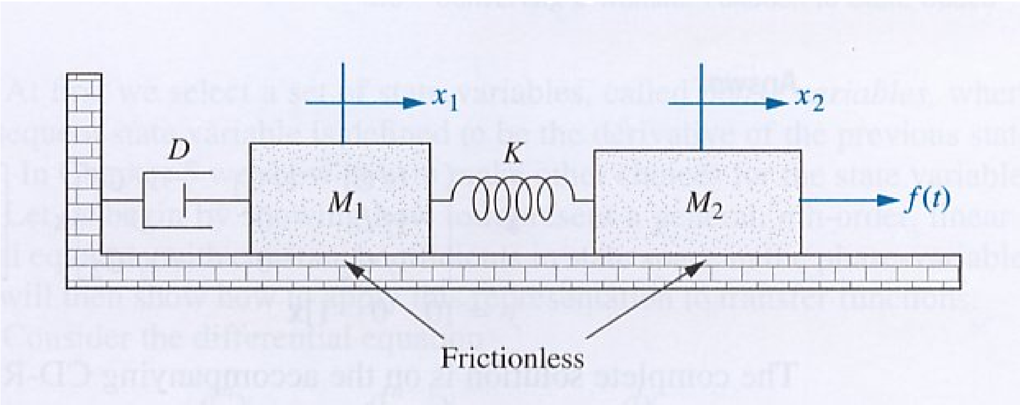
$$
\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t)
$$

The output equation is

$$
i_R = [1/R \;\; 0] \begin{bmatrix} v_C \\ i_L \end{bmatrix}
$$

Mechanical Example

Develop a set of state equations for a mechanical system below.



The sum of all forces at M_1 is zero.

$$M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + K(x_1 - x_2) = 0$$

Force $f(t)$ balances the rest of all forces.

$$K(-x_1 + x_2) + M_2 \frac{d^2 x_2}{dt^2} = f(t)$$

$$
\begin{aligned}
\frac{dx_1}{dt} &= v_1 \\
\frac{dv_1}{dt} &= -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2 \\
\frac{dx_2}{dt} &= v_2 \\
\frac{dv_2}{dt} &= \frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t)
\end{aligned}
$$

Writing in vector-matrix form,

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)
$$

There are two masses and a spring which makes the system 3rd order. We use one more degree of freedom for velocity, which is independent of position. Thus, the number of state variables is 4.

Transfer function to State space

$$G(s) = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{Y(s)}{U(s)}$$

This transfer function represents a differential equation in the time domain.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u(t)$$

phase variable choice

Choose the output $y(t)$, and its $(n-1)$ derivatives as the state variables.

$$x_1 = y \quad \longrightarrow \quad \dot{x}_1 = \frac{dy}{dt}$$

$$x_2 = \frac{dy}{dt} \quad \longrightarrow \quad \dot{x}_2 = \frac{dy^2}{dt^2}$$

$$x_3 = \frac{dy^2}{dt^2} \quad \longrightarrow \quad \dot{x}_3 = \frac{dy^3}{dt^3}$$

$$\vdots \qquad\qquad\qquad \vdots$$

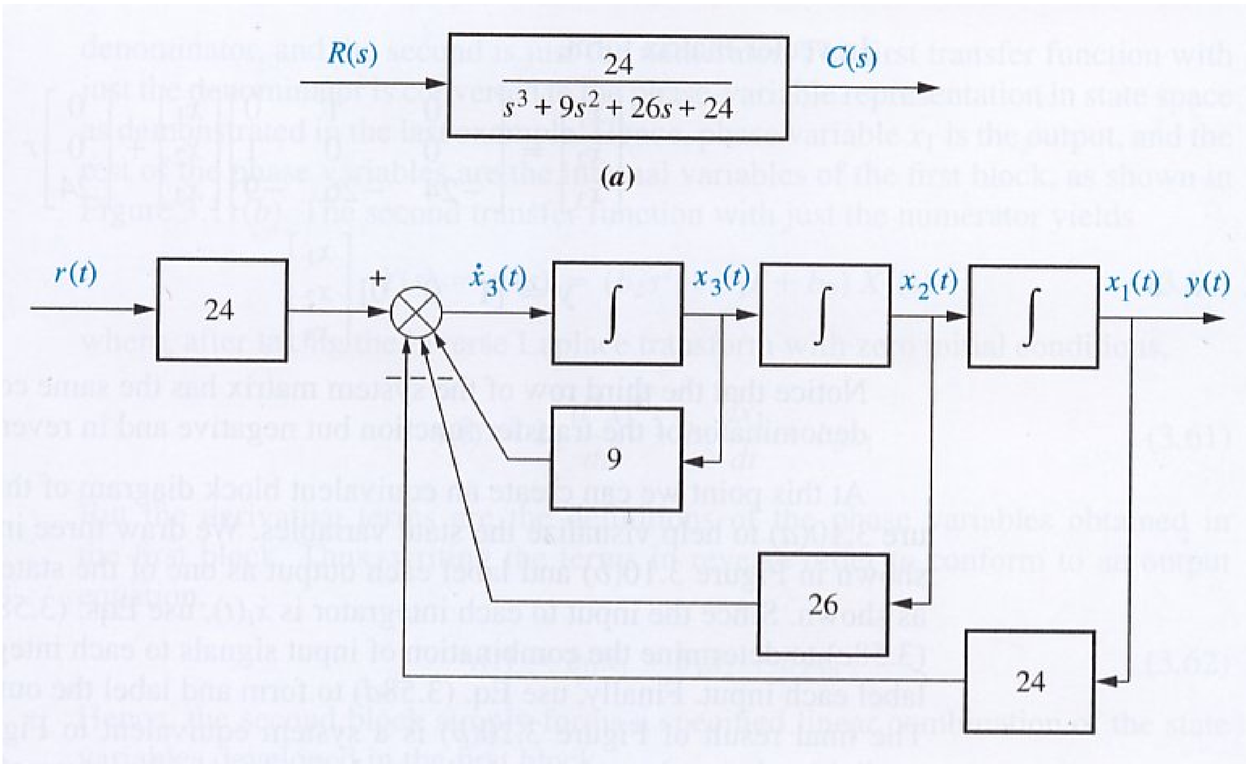
$$x_n = \frac{dy^{n-1}}{dt^{n-1}} \quad \longrightarrow \quad \dot{x}_n = \frac{dy^n}{dt^n}$$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x^{n-1} + b_0 u(t)
\end{aligned}$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u(t)
$$

$$
y = [1 \ 0 \ \cdots \ 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
$$

Example: constant in numerator



$$G(s) = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{C(s)}{R(s)}$$

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$s^3C(s) = 24R(s) - 9s^2C(s) - 26sC(s) - 24C(s)$$

$$s^3C(s) = 24R(s) - 9X_3(s) - 26X_2(s) - 24X_1(s)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r(t)$$

Example: polynomial in numerator

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24} = \frac{C(s)}{R(s)}$$

The general form is,

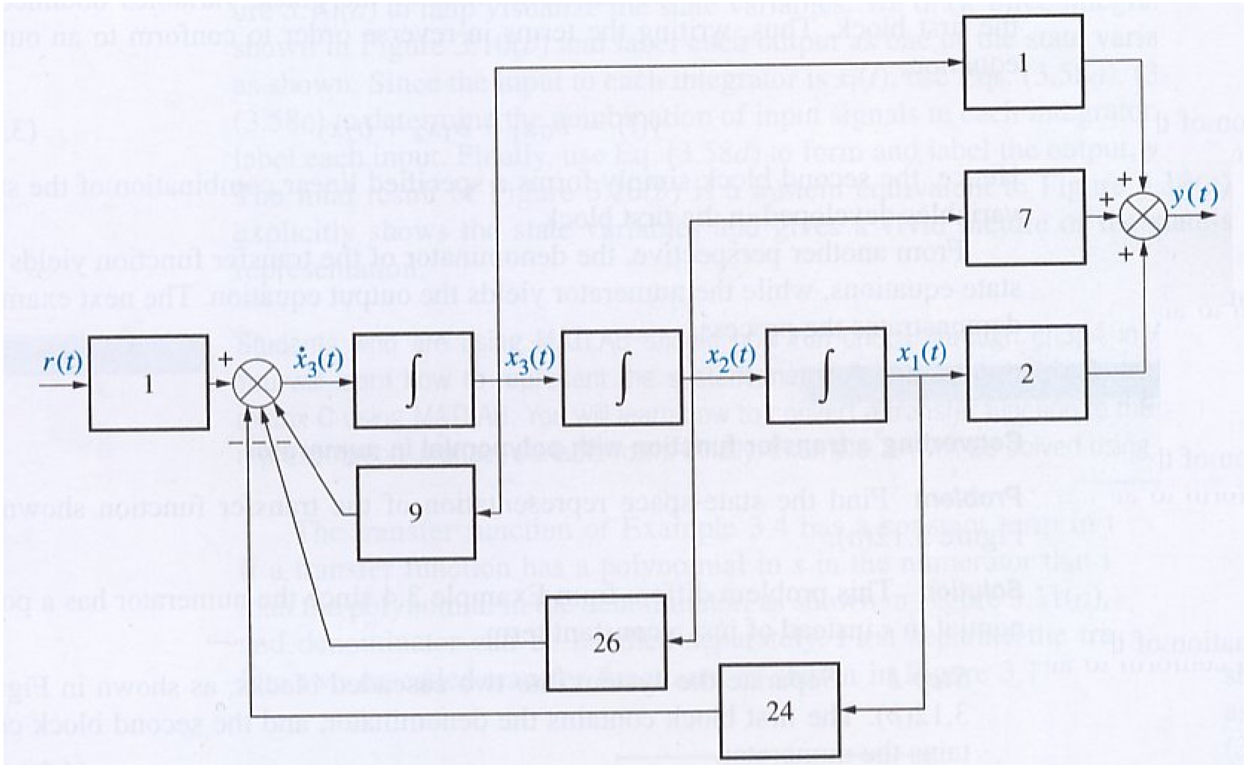
$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{C(s)}{R(s)}$$

Using an intermediate output $F(s)$,

$$\begin{aligned} F(s) &= \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} R(s) \\ C(s) &= (b_2 s^2 + b_1 s + b_0) F(s) \end{aligned}$$

$$\begin{aligned} s^3 F(s) &= R(s) - a_2 s^2 F(s) - a_1 s F(s) - a_0 F(s) \\ C(s) &= b_2 s^2 F(s) + b_1 s F(s) + b_0 F(s) \end{aligned}$$

$$\begin{aligned} s^3 F(s) &= R(s) - a_2 X_3(s) - a_1 X_2(s) - a_0 X_1(s) \\ C(s) &= b_2 X_3(s) + b_1 X_2(s) + b_0 X_1(s) \end{aligned}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$\begin{aligned} y &= c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + 7x_2 + 2x_1 \\ &= \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

State space to Transfer function

The vector/matrix form of the state equation and output equation for single input and single output is,

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y = \mathbf{c}\mathbf{x} + du \end{cases}$$

Take the Laplace transform,

$$\begin{cases} s\mathbf{X} - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{b}U(s) \\ Y(s) = \mathbf{c}\mathbf{X}(s) + dU(s) \end{cases}$$

Transfer function assumes $\mathbf{x}(0) = \mathbf{0}$.

$$\begin{cases} s\mathbf{X} - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{b}U(s) \\ Y(s) = \mathbf{c}\mathbf{X}(s) + dU(s) \end{cases}$$

$$
\begin{aligned}
s\mathbf{X} - \mathbf{A}\mathbf{X}(s) &= (s\mathbf{I} - A)\mathbf{X}(s) = \mathbf{b}U(s) \\
\mathbf{X}(s) &= (s\mathbf{I} - A)^{-1}\mathbf{b}U(s)
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= \mathbf{c}\mathbf{X}(s) + dU(s) \\
&= \mathbf{c}(s\mathbf{I} - A)^{-1}\mathbf{b}U(s) + dU(s) \\
&= \left[\mathbf{c}(s\mathbf{I} - A)^{-1}\mathbf{b} + d\right] U(s)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{Y(s)}{U(s)} &= \mathbf{c}(s\mathbf{I} - A)^{-1}\mathbf{b} + d \\
&= \mathbf{c}(s\mathbf{I} - A)^{-1}\mathbf{b}, \quad \text{if } d = 0
\end{aligned}
$$

Where,

$$
(s\mathbf{I} - A)^{-1} = \frac{\text{adj}(s\mathbf{I} - A)}{\det(s\mathbf{I} - A)}
$$

Example:

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = [1.5 \quad 0.625]\ \mathbf{x}$$

$$s\mathbf{I} - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

$$\det(s\mathbf{I} - A) = \begin{vmatrix} s+4 & 1.5 \\ -4 & s \end{vmatrix} = (s+4)s - 1.5(-4) = s^2 + 4s + 6$$

$$\mathrm{adj}(s\mathbf{I} - A) = \begin{bmatrix} s & -(-4) \\ -1.5 & s+4 \end{bmatrix}^T = \begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{c}\{\mathrm{adj}(s\mathbf{I}-A)\}\mathbf{b} &= [1.5\ 0.625]\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= [1.5\ 0.625]\begin{bmatrix} 2s \\ 8 \end{bmatrix} = 3s+5
\end{aligned}
$$

Thus,

$$
G(s) = \frac{3s+5}{s^2+4s+6}
$$