Note 13

Introduction to
Digital Control Systems
1. Introduction

A digital control system is one in which the transfer function, representing the compensator built with analog components, are now replaced with a digital computer that performs calculations that emulate the physical compensator. The following is an example of using digital control system for azimuth position control.

The structure of a typical digital controller is as follows.

![Diagram of a digital control system](image)

The signals in the above control loop take on two forms: digital or analogy. Up to this point we have used analogy signals exclusively. Digital signals, which consist of a sequence of binary numbers (e.g. 10101011), can be found in loops containing digital computers. Loops containing both analog and digital signals must provide a means for conversion from one form to the other as required by each subsystem. A device that converts analogy signals to digital signals is called an analog-to-digital (A/D) converter. Conversely, a device that converts digital signals to analog singles is called a digital-to-analog (D/A) converter.

In an A/D converter, the analog signal is sampled at a periodic interval and then held over the sampling interval by a device called a zero-order sample-and-hold (z.o.h). Samples are held before being digitized because a certain time period is required for an A/D converter to convert an analog voltage to its digital form or, in other words, the constant analog voltage must be present during the conversion process. Ideal sampling and the z.o.h. are presented in the following figure.
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\( f(t) \): Analog signal

\( f^*(t) \): Sampled waveform

\( f_{zh}(t) \): z.o.h. output

\( f^*(t) \) is the sampled waveform, consisting of the samples, \( f(kT) \). Conversion from the analog signal \( f(t) \) to the sample, \( f(kT) \), occurs repeatedly at instants of time \( T \) seconds apart. \( T \) is the **sampling interval or sampling time**, \( 1/T \) is the **sampling rate** in Hertz, and \( k \) can take on any integer value between 0 and \( +\infty \).

2. **z-Transform**

Digital control systems can be modeled adequately by the discrete equivalent to the differential equation, namely the **difference equation**. For example, the general second-order difference equation

\[
a_2 y(kT) + a_1 y(kT - T) + a_0 y(kT - 2T) = b_2 x(kT) + b_1 x(kT - T) + b_0 x(kT - 2T)
\]

where \( y \) is the system output and \( x \) is the system input.

In analog or continuous control systems, we used Laplace transforms in our analysis. In digital control systems we need to use a new transformation in order to simplify our analysis, which is called the z-**transform**. The z-transform is defined by

\[
z\{f(KT)\} = F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}
\]

**Example**

Find the z-transform of a sampled unit ramp.
The $z$-transform may be obtained by using table, much the same way as the Laplace transform. The $z$-transform conversion table is given in Table 1 and the properties of $z$-transform are provided in Table 2.

**Table 1 $z$- and s-transform**

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
<th>$F(z)$</th>
<th>$f(kT)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\frac{z}{z - 1}$</td>
<td>$u(kT)$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
<td>$\frac{Tz}{(z - 1)^2}$</td>
<td>$kT$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$\lim_{\alpha \to 0} (-1)^n \frac{d^n}{da^n} \left[ \frac{z}{z - e^{-\alpha T}} \right]$</td>
<td>$(kT)^n$</td>
</tr>
<tr>
<td>$e^{-at}$</td>
<td>$\frac{1}{s + a}$</td>
<td>$\frac{z}{z - e^{-aT}}$</td>
<td>$e^{-at}e^{-okT}$</td>
</tr>
<tr>
<td>$e^{-at}e^{-aT}$</td>
<td>$\frac{n!}{(s + a)^{n+1}}$</td>
<td>$(-1)^n \frac{d^n}{da^n} \left[ \frac{z}{z - e^{-aT}} \right]$</td>
<td>$(kT)^n e^{-at}e^{-okT}$</td>
</tr>
<tr>
<td>$\sin \omega t$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
<td>$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$</td>
<td>$\sin \omega T$</td>
</tr>
<tr>
<td>$\cos \omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
<td>$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$</td>
<td>$\cos \omega T$</td>
</tr>
<tr>
<td>$e^{-at} \sin \omega t$</td>
<td>$\frac{\omega}{(s + a)^2 + \omega^2}$</td>
<td>$\frac{ze^{-aT} \sin \omega T}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$</td>
<td>$e^{-at} \sin \omega T$</td>
</tr>
<tr>
<td>$e^{-at} \cos \omega t$</td>
<td>$\frac{s + a}{(s + a)^2 + \omega^2}$</td>
<td>$\frac{z^2 - ze^{-aT} \cos \omega T + e^{-2aT}}{z^2 - 2ze^{-aT} \cos \omega T + e^{-2aT}}$</td>
<td>$e^{-at} \cos \omega T$</td>
</tr>
</tbody>
</table>

**Table 2 z-transform theorems**

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $z[af(t)] = aF(z)$</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>2. $z[f_1(t) + f_2(t)] = F_1(z) + F_2(z)$</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>3. $z[e^{-at}f(t)] = F(e^{at}z)$</td>
<td>Complex differentiation</td>
</tr>
<tr>
<td>4. $z[f(t - nT)] = z^{-n}F(z)$</td>
<td>Real translation</td>
</tr>
<tr>
<td>5. $z[f(t)] = -Tz \frac{dF(z)}{dz}$</td>
<td>Complex differentiation</td>
</tr>
<tr>
<td>6. $f(0) = \lim_{z \to \infty} F(z)$</td>
<td>Initial value theorem</td>
</tr>
<tr>
<td>7. $f(\infty) = \lim_{z \to 1} (1 - z^{-1})F(z)$</td>
<td>Final value theorem</td>
</tr>
</tbody>
</table>

Note: $kT$ may be substituted for $t$ in the table.
In Table 2, the **Real translation theorem** tells us

\[ z\{f(KT - nT)\} = z^{-n}F(z) \]

Applying the real translation theorem to the previous general second-order difference equation, we have

\[ a_2Y(z) + a_1z^{-1}Y(z) + a_0z^{-2}Y(z) = b_2X(z) + b_1z^{-1}X(z) + b_0z^{-2}X(z) \]

The above equation then results in the discrete transfer function

\[
\frac{Y(z)}{X(z)} = \frac{b_2 + b_1z^{-1} + b_0z^{-2}}{a_2 + a_1z^{-1} + a_0z^{-2}} \quad \text{or} \quad \frac{b_2z^2 + b_1z^1 + b_0}{a_2z^2 + a_1z^1 + a_0}
\]

### 3. Controller Design via the s-Plane

There are a number of strategies or methods that could be used for the design of discrete controllers. To illustrate the implementation of digital controllers we will consider a method that allows us to design controllers via the s-Plane and then to convert the design into a discrete form.

**The Tustin transformation** is used to transform the continuous compensator, \( G_c(s) \), to the digital compensator, \( G_c(z) \). The Tustin transformation is given by

\[ s = \frac{2(z - 1)}{T(z + 1)} \]

and its inverse by

\[ z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} \]

As the sampling interval, \( T \), gets smaller (high sampling rate), the digital compensator's output yields a closer match to the analog compensator. If the sampling rate is not high enough, there is a discrepancy at higher frequencies between the digital and analog frequency responses.

**Problem**

A controller was designed with \( G_c(s) = \frac{1977(s + 6)}{(s + 29.1)} \). If the system is to be computer controlled, find the digital controller \( G_c(z) \). Use the sampling time of 0.01 second.
4. Implementing the Digital Compensator

Consider the following block diagram which may be part of a bigger control system:

The input to the digital compensator (or controller) is the sampled error signal $E(z)$, and its output is $X(z)$, which is used to drive the plant. Now we will see how to implement the digital compensator, $G_c(z)$, within a digital computer. For this, we have two steps:

**Step 1:** Derive the difference equation from the digital transfer function, by taking the inverse $z$-transform and using the inverse real translation theorem, i.e.,

$$z^{-1} \{z^{-n} F(z)\} = f(KT - nT)$$

**Step 2:** Develop a flowchart for the digital compensator based on the difference equation, and then program (e.g. using Matlab or Simulink) to realize it.

**Example**

Let’s consider a digital compensator, $G_c(z)$,

$$G_c(z) = \frac{X(z)}{E(z)} = \frac{z + 0.5}{z^2 - 0.5z + 0.7}$$
Step 1: Derive the difference equation from the digital transfer function.

Step 2: Develop a flowchart for the digital compensator based on the difference equation, and then program to realize it.

The above flowchart shows that the compensator can be implemented by storing several successive values of the input and output. The output is then formed by a weighted linear combination of these stored variables.

In Simulink, the block of ‘Unit Delay’, i.e.,

is used to perform a delay of one sample period. Thus, if using Simulink to realize the digital compensator, we can use the block of ‘Unit Delay’ and the block of ‘Gain’ to simply replace the corresponding blocks in the above flowchart so as to create a Simulink model.