Note 2

Laplace Transform & Transfer Function
1. Dynamic Systems

This course is about controlling dynamic systems. What makes dynamic systems unique and distinctive is that dynamic systems have a memory. Their present output, \( y(t) \), does not just depend on the present input signal value \( u(t) \), but on past inputs \( u(\tau) \) for \( \tau \leq t \).

The way this “memory” is described in mathematical form is through integral/differential equations. In this course, we will learn that even simple dynamic systems are modeled by derivative and integral terms.

The Laplace transform allows us to transform these integral/differential equations into simpler algebraic equations and solving them. It is a method for solving differential equations and corresponding initial and boundary value problems.

2. Review of Complex Variables and Complex Functions

Analysis of dynamic systems relies heavily on the theories associated with complex variables and complex functions. A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex number is called a complex variable. In the Laplace transformation, we use the notation \( s \) as a complex variable; that is,

\[
s = \sigma + j\omega
\]

where \( \sigma \) is the real part and \( \omega \) is the imaginary part.

**Complex function:**

A complex function \( F(s) \), a function of \( s \), has a real part and an imaginary part or

\[
F(s) = F_x + jF_y
\]

where \( F_x \) is the real part and \( jF_y \) is the imaginary part. \( F(s) \) can be represented graphically in the complex plane as follows:
The magnitude of $F(s)$ is denoted by $|F|$ and is equal to $\sqrt{F_x^2 + F_y^2}$. Its angle $\theta$ is obtained as $\tan^{-1}(F_y / F_x)$. The angle is measured counterclockwise from the positive real axis.

The complex conjugate of $F(s)$ is defined as $\bar{F} = F_x - jF_y$. Another form in which complex conjugate is denoted is $F^*$. Note that the magnitude of $F(s)$, denoted by $|F|$, can also be defined as: $|F| = \sqrt{\bar{F}F} = \sqrt{F_x^2 + F_y^2}$

3. Review of Laplace Transform

We can convert many common functions, such as sinusoidal functions, differential equations, and exponential functions, into algebraic functions of a complex variable $s$. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Thus, a linear differential equation can be transformed into an algebraic equation in a complex variable $s$. If the algebraic equation in $s$ is solved for the dependent variable, then the solution of the differential equation (the inverse Laplace transform of the dependent variable) may be found by use of a Laplace transform table or by use of the partial-fraction expansion technique.

An advantage of the Laplace transform method is that it allows the use of graphical techniques for predicting the system performance without actually solving system differential equations. Another advantage of the Laplace transform method is that, when we solve the differential equation, both the transient component and steady-state component of the solution can be obtained simultaneously.

Let us define

$f(t) = a$ function of time $t$ such that $f(t) = 0$ for $t < 0$
$s = a$ complex variable
$L = a$ operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty f(t)e^{-st} \, dt$.
$F(s) = Laplace$ transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st} \, dt$

Laplace transform of common functions have been tabulated [see the text book page 40] and are used to transform the time domain functions $f(t)$ into their Laplace form directly from tables and, vice versa.
Important Properties of Laplace Transforms

**Real differentiation theorem.**
The Laplace transform of the derivative of a function \( f(t) \) is given by

\[
L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)
\]

where \( f(0) \) is the initial value of \( f(t) \) evaluated at \( t = 0 \).

Similarly, we obtain the following relationship for the second derivative of \( f(t) \) applies:

\[
L\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - \dot{f}(0) - sf(0)
\]

and for the \( n^{th} \) derivative:

\[
L\left[\frac{d^n f(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \ldots - s^{(n-2)}\dddot{f}(0) - \ldots - f^{(n-1)}(0)
\]

In most of the problems that we will consider in this course, it is assumed that the initial values of \( f(t) \) and its derivatives are equal to zero, then the Laplace transform of the \( n^{th} \) derivative of \( f(t) \) is given by \( s^nF(s) \).

**Real-integration theorem.**

Similarly to above, the Laplace transform of \( \int_0^t f(t)dt \) exists and is given by

\[
L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s} + \frac{f(0)}{s}
\]

for zero initial value,

\[
L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}.
\]

**Final-value theorem.**

The final-value theorem is extensively used throughout the course and relates the steady-state behavior of \( f(t) \) to the behavior of \( sF(s) \) in the neighborhood of \( s = 0 \). The final-value theorem may be stated as follows:

- If \( f(t) \) and \( \frac{df(t)}{dt} \) are Laplace transformable, if \( F(s) \) is the Laplace transform of \( f(t) \), and if \( \lim_{t \to \infty} f(t) \) exists, then

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]
This formula allows you to calculate the final steady state value of \( f(t) \) by using \( F(s) \) without having to convert \( F(s) \) back into its time domain representation \( f(t) \).

Other important properties of Laplace transforms are tabulated in the text book page 41.

**Inverse Laplace Transformation**

The reverse process of finding the time function \( f(t) \) from the Laplace transform \( F(s) \) is called the inverse Laplace transformation. The notation for the inverse Laplace transform is \( L^{-1} \), and the inverse Laplace transform can be found from \( F(s) \) by the following inversion integral:

\[
L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds \quad \text{for } t > 0
\]

where \( c \) is a constant related to the stability and convergence of the integral.

Evaluating the inversion integral appears complicated. In practice, we seldom use this integral for finding \( f(t) \). Instead, we usually decompose the \( F(s) \) into a simpler form by using partial fraction expansion and, then use a conversion table to obtain the time domain equivalent of the partial fractions.

If a particular transform \( F(s) \) cannot be found in a table, then we expand it into partial fractions and write \( F(s) \) in terms of simple functions of \( s \) for which the inverse Laplace transforms are already known.

**Partial-fraction expansion method for finding inverse Laplace transforms:**

For problems in control systems analysis, \( F(s) \), the Laplace transform of \( f(t) \), frequently occurs in the form \( F(s) = B(s)/A(s) \) where \( A(s) \) and \( B(s) \) are polynomials in \( s \). In the expansion of \( F(s) = B(s)/A(s) \) into a partial-fraction form, it is important that the highest power of \( s \) in \( A(s) \) be greater than the highest power of \( s \) in \( B(s) \). If such is not the case, the numerator \( B(s) \) must be divided by the denominator \( A(s) \) in order to produce a polynomial in \( s \) plus a remainder (a ratio of polynomials in \( s \) whose numerator is of lower degree than the denominator).

If \( F(s) \) is broken up into components, \( F(s) = F_1(s) + ... + F_n(s) \) then its inverse Laplace transform is obtained as:

\[
L^{-1}[F(s)] = L^{-1}[F_1(s)] + ... + L^{-1}[F_n(s)] = f_1(t) + ... + f_n(t)
\]
The advantage of the partial-fraction expansion approach is that the individual terms of 
\( F(s) \), resulting from the expansion into partial-fraction form, are very simple functions of 
s; consequently, it is not necessary to refer to a Laplace transform table if we memorize 
several simple Laplace transform pairs. It should be noted, however, that in applying 
the partial-fraction expansion technique in the search for the inverse Laplace transform of 
\( F(s) = \frac{B(s)}{A(s)} \) the roots of the denominator polynomial \( A(s) \) must be obtained in 
advance. That is, this method does not apply until the denominator polynomial has been 
factored.

**Partial-fraction expansion when \( F(s) \) involves distinct poles only:**

Consider \( F(s) \) written in the factored form

\[
F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1) ... (s + z_m)}{(s + p_1) ... (s + p_n)} \quad \text{for} \quad m < n
\]

where \( p \) and \( z \) are either real or complex quantities.

If \( F(s) \) involves distinct poles only, then it can be expanded into a sum of simple partial 
fractions as follows:

\[
F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + ... + \frac{a_n}{s + p_n}
\]

where \( a_k \) \( (k = 1, 2, \ldots, n) \) are constants. The coefficient \( a_k \) is called the *residue* at the 
pole at \( s = -p_k \). The value of \( a_k \) can be found by multiplying both sides of the above 
Equation by \( (s + p_k) \) and letting \( s = -p_k \) which gives:

\[
a_k = \left[ (s + p_k) \frac{B(s)}{A(s)} \right]_{s=-p_k}
\]

since: 

\[
L^{-1}\left[ \frac{a_k}{s + p_k} \right] = a_k e^{-p_k t} \quad \text{then} \quad f(t) \quad \text{is obtained as} \quad f(t) = a_1 e^{-p_1 t} + ... + a_n e^{-p_n t} \quad \text{for all} \quad t \geq 0.
\]

**4. Transfer Functions**

A mathematical model of a dynamic system is defined as a set of equations that 
represents the dynamics of the system accurately or, at least, fairly well. Note that a 
mathematical model is not unique to a given system. A system may be represented in
many different ways and, therefore, may have many mathematical models, depending on one's perspective. The degree of accuracy and complexity of models is very much dependent on the use and application that they are intended for.

The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular system, for example, Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems. We must always keep in mind that deriving a reasonable mathematical model is the most important part of the entire analysis.

Mathematical models
Mathematical models may assume many different forms. We will consider considering the transient-response or frequency-response analysis of single-input-single-output, linear, time-invariant systems. Here, a transfer function representation is more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computer tools can be used for analysis and synthesis purposes.

Simplicity versus accuracy
It is possible to improve the accuracy of a mathematical model by increasing its complexity. In some cases, we include hundreds of equations to describe a complete system at a certain level of accuracy. If extreme accuracy is not needed, however, it is preferable to obtain only a reasonably simplified model. In fact, we are generally satisfied if we can obtain a mathematical model that is adequate for the problem under consideration. It is important to note, however, that the results obtained from the analysis are valid only to the extent that the model approximates a given dynamic system.

In general, in solving a new problem, we find it desirable first to build a simplified model so that we can get a general feeling for the solution. A more complete mathematical model may then be built and used for a more complete analysis.

Linear systems
A system is called linear if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses. Hence, for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results. It is this principle that allows one to build up complicated solutions to the linear differential equation from simple solutions.

Linear time-invariant systems and linear time-varying systems
A differential equation is linear if the coefficients are constants or functions only of the independent variable. Dynamic systems that are composed of linear time-invariant lumped-parameter components may be described by linear time-invariant (constant coefficient) differential equations. Such systems are called linear time-invariant (or linear constant-coefficient) systems. Systems that are represented by differential equations whose coefficients are functions of time are called linear time-varying systems.
An example of a time-varying control system is a spacecraft control system. (i.e. mass of a spacecraft changes due to fuel consumption.)

**Nonlinear systems**

A system is nonlinear if the principle of super-position does not apply. Thus, for a nonlinear system the response to two inputs cannot be calculated by treating one input at a time and adding the results. Although many physical relationships are often represented by linear equations, in most cases actual relationships are not quite linear. In fact, a careful study of physical systems reveals that even so-called "linear systems" are really linear only in limited operating ranges. In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve nonlinear relationships among the variables. For example, the output of a component may saturate for large input signals. There may be a dead space that affects small signals. (The dead space of a component is a small range of input variations to which the component is insensitive). Square-law nonlinearity may occur in some components. For instance, dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities, and the damping force may become proportional to the square of the operating velocity. Examples of characteristic curves for these nonlinearities are shown in your text.

**Transfer Functions**

In control theory, functions called transfer functions are commonly used to characterize the input-output relationships of components or systems that can be described by linear, time-invariant, differential equation.

The *transfer function* of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero. Consider the linear time-invariant system defined by the following differential equation:

\[
\begin{align*}
\sum_{i=0}^{n} a_i y^{(i)} + \sum_{j=0}^{n-1} a_j y^{(j)} + a_n y & = \sum_{i=1}^{m} b_i x^{(i)} + b_m x \\
& = b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m \\
& + \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}
\end{align*}
\]

where \( y \) is the output of the system and \( x \) is the input. The transfer function of this system is obtained by taking the Laplace transforms of both sides of the above equation, under the assumption that all initial conditions are zero.

\[
\text{Transfer function } G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}
\]

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in \( s \). If the highest power of \( s \) in the denominator of the transfer function is equal to \( n \), the system is called an *nth-order system.*
Comments on transfer function
The applicability of the concept of the transfer function is limited to linear, time-
invariant, differential equation systems. The transfer function approach, however, is
extensively used in the analysis and design of such systems. In what follows, we shall
list important comments concerning the transfer function. (Note that in the list a system
referred to is one described by a linear, time invariant, differential equation.)

1. The transfer function of a system is a mathematical model in that it is an operational
   method of expressing the differential equation that relates the output variable to the
   input variable.
2. The transfer function is a property of a system itself independent of the magnitude
   and nature of the input or driving function.
3. The transfer function includes the units necessary to relate the input to the output;
   however, it does not provide any information concerning the physical structure of the
   system. (i.e. transfer functions of many physically different systems can be
   identical.)
4. If the transfer function of a system is known, the output or response can be studied
   for various forms of inputs with a view toward understanding the nature of the
   system.
5. If the transfer function of a system is unknown, it may be established experimentally
   by introducing known inputs and studying the output of the system. Once
   established, a transfer function gives a full description of the dynamic characteristics
   of the system, as distinct from its physical description.

To derive the transfer function, we proceed according to the following steps.
1. Write the differential equation for the system.
2. Take the Laplace transform of the differential equation, assuming all initial
   conditions are zero.
3. Take the ratio of the Laplace transforms of the output to the input. This ratio is
   the transfer function.