Note 8

Root-Locus Techniques
1. Vector Representation of Complex Numbers

A complex number, denoted by $z$, can be represented in the complex plane by using a vector, as shown in the following figure. The length of the vector is the magnitude of the complex number and the angle measured counterclockwise from the real axis is the angle of the complex number.

![Vector Representation of Complex Numbers](image)

Suppose that both $p$ and $s$ are complex numbers. The subtraction of $(s-p)$ is also a complex number, which can be represented by a vector drawn from the location of $p$ to the location of $s$, as shown in the following figure. If $s$ is varied, then vector $(s-p)$ changes in both magnitude and angle.

![Vector Representation of Complex Numbers](image)

**Example**

If $s = -3 + 4j$, represent the vectors of $s+1$, $s$, and $s+2$ in the complex plane, and then evaluate the magnitude and angle of $F(s) = \frac{(s+1)}{s(s+2)}$. 

$|z| = |s| = \sqrt{(-3)^2 + 4^2} = 5$

$\theta = \tan^{-1}\left(\frac{4}{-3}\right) = -53.13°$
2. Root-Locus

Definition: The root locus is the plot of the roots of system characteristic equation (or the poles of closed-loop transfer function) as a system parameter ($K$) is varied.

Consider the following closed loop system:

\[
\begin{align*}
R(s) & \quad + \\
K & \quad 1 \\
\frac{1}{s(s+2)} & \quad C(s)
\end{align*}
\]

The closed loop transfer function is obtained as:

\[
T(s) = \frac{KG(s)}{1 + KG(s)}
\]

For \( G(s) = \frac{1}{s(s+2)} \), then

\[
T(s) = \frac{K}{s^2 + 2s + K}
\]

The characteristic equation is obtained as:

\[
s^2 + 2s + K = 0
\]

The poles can be found to be:

\[
s = \frac{-2 \pm \sqrt{4 - 4K}}{2} = -1 \pm \sqrt{1 - K}
\]
It can be seen from the above equation that the poles change as the value of $K$ is varied. The change in $K$ therefore affects the stability and the performance of the system.

For the above second order system, one has

$$\omega_n = \sqrt{K}$$

$$\zeta = 2 / 2\sqrt{K}$$

In second order systems, the effect of $K$ on the natural frequency and damping ratio can be readily determined. For higher order systems, this analysis is not as obvious. This is where the root locus design method is used.

In the above example, the change of poles with increase in $K$ can be obtained and plotted as:

<table>
<thead>
<tr>
<th>$K$</th>
<th>Pole 1</th>
<th>Pole 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-1+j 1.00</td>
<td>-1-j 1.00</td>
</tr>
<tr>
<td>3</td>
<td>-1+j 1.41</td>
<td>-1-j 1.41</td>
</tr>
<tr>
<td>4</td>
<td>-1+j 1.73</td>
<td>-1-j 1.73</td>
</tr>
<tr>
<td>5</td>
<td>-1+j 2.00</td>
<td>-1-j 2.00</td>
</tr>
<tr>
<td>10</td>
<td>-1+j 3.00</td>
<td>-1-j 3.00</td>
</tr>
<tr>
<td>20</td>
<td>-1+j 4.36</td>
<td>-1-j 4.36</td>
</tr>
<tr>
<td>30</td>
<td>-1+j 5.39</td>
<td>-1-j 5.39</td>
</tr>
<tr>
<td>40</td>
<td>-1+j 6.24</td>
<td>-1-j 6.24</td>
</tr>
<tr>
<td>50</td>
<td>-1+j 7.00</td>
<td>-1-j 7.00</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

Calculation of the roots, given a change in gain, can be readily accomplished by using design packages such as MATLAB. However, the root locus may also be sketched by hand by following the procedure that is described in the following sections.
3. Properties of the Root-Locus

Consider the following closed-loop system

The closed-loop transfer function is

\[ T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \]

The characteristic equation:

\[ 1 + KG(s)H(s) = 0 \]

Or

\[ KG(s)H(s) = -1 \]

This in turn requires:

1. **Magnitude property**: the magnitude of the open-loop transfer function is 1, i.e.,

   \[ |KG(s)H(s)| = 1 \quad \text{or} \quad K = \frac{1}{|G(s)H(s)|} \]

2. **Angle property**: the angle of the open-loop transfer function is an odd multiple of 180°, i.e.,

   \[ \angle(KG(s)H(s)) = n \times 180° \quad \text{for} \quad n = \pm 1, \pm 3, \pm 5, \ldots \]

   or \[ \angle(G(s)H(s)) = n \times 180° \quad \text{for} \quad n = \pm 1, \pm 3, \pm 5, \ldots \]
Example

Suppose the open-loop transfer function \( KG(s)H(s) = \frac{K}{s(s + 2)} \)

(1) Are \( s = -1 + j \) and \( s = 3j \) on the root locus?
(2) If so, what is the corresponding value of \( K \)?

Finite and Infinite the Open-Loop Poles and Zero

Consider the following open-loop transfer function:

\[
KG(s)H(s) = \frac{K(s + 2)}{(s + 1)(s + 20)}
\]

In this example, there are 2 finite poles at -1 and -20 and one finite zero at -2.

Besides, a function could have infinite poles and zeros. If the function approaches infinity as \( s \) goes infinity, then the function has pole(s) at infinity. If the function approaches zero as \( s \) approaches infinity, it has zero(s) at infinity. For the above example, as \( s \) approaches infinity:

\[
\lim_{s \to \infty} KG(s)H(s) = \lim_{s \to \infty} \frac{K(s + 2)}{(s + 1)(s + 20)} = 0
\]

Thus, the function has an infinite zero.

Please not that every function of \( s \) has an equal number of poles and zeros, including both infinite and finite poles and zeros. In the above example, the function has two finite poles, one finite zero and one infinite zero, thus

the number of poles = the number of zeros = 2
4. Sketching the Root Locus

Before digital computers, sketching of the root locus was performed by taking advantage of the angle and magnitude criteria. We will review this method to gain further insight into root locus. When working in industry, you are strongly advised to use design tools such as Matlab.

The following rules facilitate the sketching of the root locus:

1. **Number of Branches:** The number of branches of root locus equal to the number of closed loop poles. There will be one branch for each closed loop pole.

2. **Symmetry:** The root locus is symmetrical with respect to the real axis.

3. **Real-axis Segments:** On the real-axis the root locus exists only in the segment where the number of finite open-loop poles and zeros that are located on the right-hand side of the segment is odd.

Further to the angle criterion, a root on the real-axis must satisfy the following requirement:

\[ \Sigma \text{angles from finite zeros} - \Sigma \text{angles from finite poles} = n \times 180^\circ \quad \text{for} \quad n = \pm 1, \pm 3, \ldots \]

where \( \Sigma = \text{sum} \).

If we are considering a point on the real axis, at this point the following can be observed

1. The angular contribution of a pair of open-loop complex poles or zeros is zero.
2. The angular contribution of the open-loop poles and zeros on the real-axis, which are located on the left-hand side of the point, is also zero.

So the contribution to the angle only comes from the open-loop poles and zeros on the real-axis, which are located on the right-hand side of the point considered. The angular contribution is \( 180^\circ \) from each of such open-loop poles and zeros. If the number of these open-loop poles and zeros is odd, the point is therefore on the root locus. Otherwise, it is not on the root locus.

4. **Starting and ending points:** The root locus originates at the poles of \( G(s)H(s) \) and terminates at the zeros of \( G(s)H(s) \), including those zeros at infinity.

The explanation for this rule is as follows. The characteristic equation for the system may be expressed in the form:
This equation may be rearranged and rewritten as:

\[ \prod_{j=1}^{m} (s - p_j) + K \prod_{i=1}^{n} (s - z_i) = 0 \]

Thus for \( K=0 \) the roots of the characteristic equation are simply the poles of \( G(s)H(s) \). As \( K \) approaches infinity but \( s \) remains finite, the branches of the locus approach the zeros of the open-loop function. If the open-loop function has zeros at infinity, that is, if \( n>m \) (as is the usual case with the models of physical systems) the locus will also approach these zeros. This aspect of the root locus will be proved in the discussion of one of the additional rules shortly.

5. Asymptotes: The root locus approaches straight lines as asymptotes as the locus approaches infinity.

Let the general form of an open-loop function be rewritten in the form:

\[ K\prod_{i=1}^{n}(s - z_i) \]

\[ 1 + KG(s)H(s) = 1 + \frac{\prod_{i=1}^{m}(s - p_j)}{\prod_{j=1}^{m}(s - p_j)} = 0 \]

For most systems considered, \( m<n \) and the above equation may be written as:

\[ KG(s)H(s) = \frac{K(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_o)}{(s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o)} \]

where \( \alpha = n-m \), is the number of infinite zeros.

\[ \lim_{s \to \infty} KG(s)H(s) = \lim_{s \to \infty} \frac{K b_m s^m}{s^n} = \lim_{s \to \infty} \frac{K b_m}{s^{\alpha}} \]

The root locus for large values of \( s \) then satisfies the relationship:

\[ \lim_{s \to \infty}[1 + KG(s)H(s)] = \lim_{s \to \infty}[1 + \frac{K b_m}{s^{\alpha}}] = 0 \]

This equation has roots that are given by
The magnitude of these roots approaches infinity because of the assumption that \( s \) is approaching infinity. The angle \( \theta \) contributed by each root is therefore:

$$\theta = \frac{r180}{\alpha}$$

These angles are the angles of asymptotes of the root locus, since generally the root loci approach these angles as \( s \) (and thus \( K \)) approaches infinity.

If we denote the value of \( s \) at which the asymptotes intersect the real axis as \( \sigma_a \), this value is given by

$$\sigma_a = \frac{(\text{sum of finite poles}) - (\text{sum of finite zeros})}{(\text{number of finite poles}) - (\text{number of finite zeros})}$$

6. **Breakaway and Break-in Points:** This rule concerns points at which two or more branches of the locus come together or break away. Points of this type are called breakaway and break-in points.

Intuitively, when two closed loop pole move towards each other along the real axis, they meet at the maximum value of the gain that would lead to real poles. The breakaway point therefore occurs at a point of maximum gain on the real axis.

Similarly, for the break in point, complex pairs of poles return to the real-axis and, as the gain increases to infinity, the poles move towards the open loop zeros. Therefore, the gain at the break-in point is the minimum gain long the real-axis between two zeros. We would therefore be able to find these minimum and maximum points by using differential calculus as follows. For an open loop transfer function \( G(s)H(s) \), it can be shown that (Philips & Harbor, 2000) the breakaway/break-in points on a root locus will appear among the roots of the polynomial:

$$\frac{d[G(s)H(s)]}{ds} = 0$$

or equivalently,

$$N'(s)D(s) - N(s)D'(s) = 0$$

where \( N(s) \) and \( D(s) \) are the numerator and denominator polynomials of \( G(s)H(s) \).
The following examples will be given in class

**Example 1**

Sketch the root locus for the system with an open-loop transfer function given by

\[ KG(s)H(s) = \frac{K(s - 3)(s - 5)}{(s + 1)(s + 2)}. \]

**Example 2**

Find the breakaway and break-in points for the system given in Example 1.

**Example 3**

Sketch the root locus for the system with an open-loop transfer function given by

\[ KG(s)H(s) = \frac{K}{(s - 1)(s + 2)(s + 3)}. \]

**Example 4**

Sketch the root locus for the system with the following open-loop transfer function given by

\[ KG(s)H(s) = \frac{K}{(s + 1)(s + 3)(s + 10)}. \]

And then determine where the root locus crosses the imaginary axis.