Note 2

Electrical Circuits
1. Basics of Linear Circuits

Basic passive components of electrical circuits are resistor, capacitor, and inductor. By passive, we mean there is no internal source of energy. The following table summarizes these passive components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

<table>
<thead>
<tr>
<th>Component</th>
<th>Voltage-current</th>
<th>Current-voltage</th>
<th>Voltage-charge</th>
<th>Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacitor</td>
<td>( v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau )</td>
<td>( i(t) = C \frac{dv(t)}{dt} )</td>
<td>( v(t) = \frac{1}{C} q(t) )</td>
<td>( \frac{1}{Cs} )</td>
</tr>
<tr>
<td>Resistor</td>
<td>( v(t) = R i(t) )</td>
<td>( i(t) = \frac{1}{R} v(t) )</td>
<td>( v(t) = R \frac{dq(t)}{dt} )</td>
<td>( R )</td>
</tr>
<tr>
<td>Inductor</td>
<td>( v(t) = L \frac{di(t)}{dt} )</td>
<td>( i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau )</td>
<td>( v(t) = L \frac{dq(t)}{dt^2} )</td>
<td>( LS )</td>
</tr>
</tbody>
</table>

where \( v(t) \) has a unit of volts, \( i(t) \) amps, \( q(t) \) coulombs, \( C \) farads, \( R \) ohms, \( H \) henries.

**Kirchhoff’s Current Law:** At any node, the sum of the currents that enter the node is equal to the sum of the currents that leave from the node.

**Kirchhoff’s Voltage Law:** The sum of all voltage drops around a closed loop is zero.

**Transfer Functions**

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero. Consider the linear time-invariant system defined by the following differential equation:

\[
a_o y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y + a_0 y = b_o x^{(m)} + b_{m-1} x^{(m-1)} + \ldots + b_1 x + b_0 x
\]

where \( y \) is the output of the system and \( x \) is the input. The transfer function of this system is obtained by taking the Laplace transforms of both sides of the above equation, under the assumption that all initial conditions are zero.

\[
\text{Transfer function: } G(s) = \frac{b_o s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_o s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}
\]

By using the concept of transfer function, it is possible to represent system dynamics by algebraic equations in \( s \). If the highest power of \( s \) in the denominator of the transfer function is equal to \( n \), the system is called an \( n \)-th order system.
Example 1

Find the transfer function relating the capacitor voltage, $V_c(s)$, to the input voltage, $V(s)$, in the following figure.

![Figure 1](image.png)

2. Impedance

The term *impedance* is a generalization of the resistance. Mathematically, the impedance of an electric component (i.e., a resistor, capacitor, or inductor) is defined as the ratio of the Laplace transform of the voltage drop of the component to the Laplace transform of the current going through the component, i.e.,

$$Z(s) = \frac{V(s)}{I(s)}$$

Notice that the above definition is similar to the definition of resistance, i.e., the ratio of voltage to current. But, unlike resistance, this function is applicable to capacitors and inductors, and also carries information on the dynamic behavior of the component since it represents an equivalent differential equation.

For a capacitor: $Z(s) = \frac{1}{C_s}$ or $Z(j\omega) = \frac{1}{Cj\omega}$

For a resistor: $Z(s) = R$ or $Z(j\omega) = R$

For an inductor: $Z(s) = LS$ or $Z(j\omega) = Lj\omega$
For a circuit that consists of several electric components (including resistors, capacitors, and inductors), we can evaluate the equivalent impedance of the circuit in the same way as we evaluate the equivalent resistance for a circuit that consists of only resistors.

(1) Components in series:

\[ Z_{\text{equiv}} = Z_1 + Z_2 \]

(2) Components in parallel:

\[ Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} \]

By means of impedance, we can easily find the transfer function of a circuit, rather than writing the differential equation first and then taking the Laplace transform. We summarize the steps as follows.

(1) Draw the transformed circuit, which is the original circuit with all time variables (e.g. \(v(t)\) and \(i(t)\)) replaced with their Laplace transform (e.g. \(V(s)\) and \(I(s)\)) and all electric component values replaced with their impedances values, respectively. For example, the following figure shows the transformed circuit given in Figure 1.

(2) Apply \( Z(s) = \frac{V(s)}{I(s)} \) to the circuit or part of the circuit to obtain the Laplace transform of the differential equation.

(3) Find the transfer function of the circuit.

\[ V(s) + \frac{1}{Cs} \rightarrow I(s) \]

\[ I(s) \rightarrow \frac{1}{Cs} \rightarrow V_c(s) \]

**Figure 2: Transformed circuit of the circuit in Figure 1**
3. Thevenin’s Equivalent Circuits

Any section of a linear circuit with multiple resistors plus voltage and current sources can be replaced with a circuit that consists of a voltage source in series with a resistor. This circuit is called the Thevenin’s equivalent circuit.

The following are the steps to find the Thevenin’s equivalent circuit shown in Figure 3.

1. Identify the subcircuit whose Thevenin’s equivalent circuit is to be determined. Identify the two point terminals (a and b) out of the subcircuit and then remove the load resistor between the two points (Figure 3(a)).

2. Calculate the open-circuit voltage between a and b, i.e., \( V_{ab} \) (Figure 3(b)).

3. Turn off (or short-circuit) all other voltage sources and disconnect (or open-circuit) all current sources, and calculate the equivalent resistor in the circuit (excluding \( R_L \)), i.e., \( R_{eqv} \) (Figure 3(c)).

4. The circuit inside the dotted box can be viewed as a voltage source (\( V_{ab} \)) plus a series resistance (\( R_{eqv} \)) (Figure 3(d)).

Example 2

Find the current though \( R_L \) for \( V_s = 10 \text{ V} \), \( R_1 = 7 \text{ }\Omega \), \( R_2 = 3 \text{ }\Omega \), \( i_s = 5 \text{ A} \) in Figure 3.
Example 3

The following figure show the DC Wheatstone bridge, which is commonly used in instrumentation to convert a change in the resistance (or other impedances) of a sensor to a change in voltage. Find the current though $R_L$ for $V_R=6\ V$, $R_1=R_2=R_4=R_5=100\Omega$, $R_3=200\ \Omega$.

Figure 4: DC Wheatstone bridge with load

4. Input Impedance and Output Impedance of an Amplifier

Electronic circuits consist of connections between components where the output of one component is connected to the input of another component. Consider the circuit in Figure 5 (a), in which an operational amplifier (op-amp) is connected to an input source (i.e., a sensor signal) and an output load. The input impedance of the amplifier is the generalized resistance seen by the input signal source, denoted by $R_{in}$ in Figure 5 (b). The output impedance of the amplifier is the generalized resistance of its equivalent Thevenin’s circuit seen by the load, denoted by $R_{out}$ in Figure 4 (b).
The input and output impedances of an amplifier are of importance to the transmitted signal and causes the *input and output loading errors*. Referring to Figure 4, we have:

\[
\text{the input voltage: } v_{\text{in}} = \frac{R_{\text{in}}}{R_{\text{in}} + R_{\text{s}}} v_s; \quad \text{and the output voltage: } v_{\text{out}} = \frac{R_{\text{L}}}{R_{\text{L}} + R_{\text{out}}} K_{\text{amp}} v_{\text{in}},
\]

where \( K_{\text{amp}} \) is the amplifier gain. The *input loading error* is the difference between \( v_{\text{in}} \) and \( v_s \); and the *output loading error* is the difference between \( (K_{\text{amp}} v_{\text{in}}) \) and \( v_{\text{out}} \). Ideally, \( R_{\text{in}} \to \infty \) and \( R_{\text{out}} \to 0 \), then \( v_{\text{in}} = v_s \) and \( v_{\text{out}} = K_{\text{amp}} v_{\text{in}} \), i.e., there is no input and output loading error.

**Example 2**

Consider the circuit in Figure 5, in which an operational amplifier (op-amp) is connected to a sensor signal and an output load. The sensor provides a voltage \( v_s(t) = 10 \text{ V} \) and the amplifier has a gain \( K_{\text{amp}} = 1.0 \). Determine the voltage, \( v_{\text{out}} \), measured by means of a digital voltmeter or an oscilloscope under the following two conditions:

1. \( R_s = 100 \Omega, \quad R_L = 100 \Omega, \quad R_{\text{in}} = 100 \Omega, \quad \text{and } R_{\text{out}} = 100 \Omega \)
2. \( R_s = 100 \Omega, \quad R_L = 100 \Omega, \quad R_{\text{in}} = 1,000,000 \Omega, \quad \text{and } R_{\text{out}} = 1 \Omega \)