Maximum Power Transfer Theorem

Preamble

- Given a circuit, how can we determine the maximum power that we can get from it (say from a car audio amplifier).

Maximum Power Transfer Theorem

- A load will receive maximum power from a linear bilateral dc network when its total resistive value is exactly equal to the Thévenin resistance of the network as “seen” by the load.

Consider the following circuit:

- This is nothing more than a Thévenin equivalent circuit with a variable load resistance, $R_L$, attached to the two terminals.
Mathematical Necessities

- For the circuit shown
  \[ I_L = \frac{E_{TH}}{R_{TH} + R_L} \]

- and therefore
  \[ P_L = I_L^2 R_L = \frac{E_{TH}^2}{(R_{TH} + R_L)^2} R_L = E_{TH}^2 (R_{TH} + R_L)^2 \]

Substituting
\[ x = R_L \]

and using the following relationship (Product Rule)
\[ [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \]

take the derivative of \( P_L(x) \) and set it equal to 0:

\[ \frac{dP_L(x)}{dx} = 1E_{TH}^2 (R_{TH} + x)^2 - 2E_{TH}^2 x(R_{TH} + x)^3 = 0 \]
\[ = E_{TH}^2 [(R_{TH} + x)^2 - 2x(R_{TH} + x)^3] = 0 \]

Simplifying
\[ 1 - \frac{2x}{R_{TH} + x} = 0 \]
\[ R_{TH} + x = 2x \]
\[ R_{TH} = x \]

Therefore (after back substituting \( R_L = x \))
\[ R_L = R_{TH} \]

The load resistance is equal to the Thévenin resistance for maximum power transfer.
Example 1
Maximum Power

What is the value for the maximum power that can be transferred to the load?

\[ V_{R_L} = \frac{E_{TH}R_L}{R_L + R_{TH}} = \frac{E_{TH}R_L}{2R_L} = \frac{E_{TH}}{2} \]

\[ I_{R_L} = \frac{E_{TH}}{R_L + R_{TH}} = \frac{E_{TH}}{2R_L} \]

\[ P_{R_L} = V_{R_L}I_{R_L} = \left( \frac{E_{TH}}{2} \right) \left( \frac{E_{TH}}{2R_L} \right) = \frac{E_{TH}^2}{4R_L} \]

Example 1
Solution:

\( (E_{th}^2/4R_L) \)

Simulation

Using Microsoft Excel you can calculate the power delivered to the load as a function of the load and plot the results.

The location of the power peak should correspond to a \( R_L \) value equal to \( R_{th} \).
The following Thévenin equivalent circuit was used:

![Thévenin equivalent circuit](image)

The following is a graph from Excel showing a plot of the power delivered to the load as the resistance is varied.

The peak of 12.5 watts occurs at $R_L = 2\Omega$.

Efficiency

The efficiency, $\eta$, in percent (%) of power transferred from circuit source(s) to a load is defined as

$$\eta = \frac{P_L}{P_S} \times 100\%$$

where

- $P_L$ is the power delivered to the load.
- $P_S$ is the power supplied by the source(s).
Maximum Power Transfer Theorem

Substituting

\[ \eta = \frac{I^2R_L}{I^2R_{\text{total}}} \times 100\% = \frac{R_L}{R_{\text{th}} + R_L} \times 100\% \]

Note:
- \( R_L >> R_{\text{th}} \) \( \eta \rightarrow 100\% \)
- \( R_L << R_{\text{th}} \) \( \eta \rightarrow 0\% \)
- \( R_L = R_{\text{th}} \) \( \eta \rightarrow 50\% \)

The following is a graph from Excel showing a plot of the efficiency of the power delivered to the load as the load resistance, \( R_L \), is varied for the circuit considered previously.

Note that at the maximum power transfer point, \( R_L = 2\ \Omega \), only 50% of the power sourced is delivered to the load.

Example 1

Maximum Power Transfer

Find the maximum power that can be transferred to the resistor \( R \).
Example 1
Solution: (16W)

- In order to do this question we first have to find the value of R (using Thévenin’s Theorem) and then apply the Maximum Power Transfer Theorem.
- In doing Thévenin’s Theorem, I will first do a source transformation of the voltage source and the $3\Omega$ resistor. The circuit then becomes

R_{th}:

- In order to calculate $R_{th}$ all sources must be zeroed. In order to zero a current source it must be open-circuited while in order to zero a voltage source it must be short circuited. With the current sources open-circuited and the load resistor, R, removed the circuit then becomes:

Therefore

$$R_{th} = 2\Omega + \frac{3\Omega}{6\Omega} = 2\Omega + 2\Omega = 4\Omega$$
and (for Maximum Power Transfer) \[ R = R_{\text{th}} = 4\Omega \]

\[ \mathbb{E}_{\text{th}}: \]
- In order to calculate \( \mathbb{E}_{\text{th}} \) the output must be open circuited (i.e., the load resistor \( R \) must be removed) so that \( V_{\text{dc}} \) can be determined. The circuit then becomes:

Note that the two current sources have a constant current, regardless of the voltage across them. The 3\( \Omega \) resistor and the 6\( \Omega \) resistor can be combined (as previously done) into an equivalent 2\( \Omega \) resistor. The circuit then becomes as below (with the current flow and polarities as indicated):
Based upon this, the voltage at the open circuit output is

\[ E_{TH} = V_{oc} = (2\Omega \cdot 6.4) + (2\Omega \cdot 2.4) = 16V \]

Therefore the maximum power that can be dissipated by the resistor is

\[ P_R = \frac{V^2}{4R} = \frac{(16V)^2}{4 \cdot 4\Omega} = 16W \]